APMA 2130 Notes
Ordinary Differential Equations

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Contents

1 First-Order Ordinary Differential Equations

Solving ordinary differential equations (ODEs) involves solving for functions that satisify a given differential equation, and importantly, for right now, we aren't considering any initial or boundary conditions. We will discuss those so-called "initial-value problems" in the next section.

1.1 Separation of Variables

We know from Calculus that we can perform the following integrals:

$$\int f(x)dx \text{ and } \int g\left(y(x)\right)dy.$$

So if we are given

$$\frac{dy}{dx}g\left(y(x)\right) - f(x) = 0,$$

we can perform a little bit of algebra to produce

$$\int g\left(y(x)\right)dy = \int f(x)dx,$$

leading us to at least an implicit solution. We may also get equations in a form similar to this:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

We can easily solve these equations by isolating terms on each side of the equation and integrating. The types of equations that can have terms separated on both sides of the equation are known as *separable equations*. Separable equations may be integrated via separation of variables, as the name suggests. Let's examine some examples:

$$\begin{aligned}
\frac{dy}{dt} + \alpha y(t) = 0& \\
\frac{dy}{dt} &= -\alpha y(t) \\
\frac{dy}{y(t)} &= -\alpha dt \\
\ln|y(t)| &= -\alpha t + c_1 \\
|y(t)| &= c_2 \exp\{-\alpha t\} \\
y(t) &= c \exp\{-\alpha t\} \qquad (1.1)
\end{aligned}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{3x^2+4x+2}{2\left(y-1\right)} \\
2\left(y-1\right)dy &= \left(3x^2+4x+2\right)dx \\
2\int\left(y-1\right)dy &= \int\left(3x^2+4x+2\right)dx \\
y^2-2y &= x^3+2x^2+2x+c_1 \\
\left(y-1\right)^2-1 &= x^3+2x^2+2x+c_1 \\
y-1 &= \pm\sqrt{x^3+2x^2+2x+c_2} \\
y &= 1\pm\sqrt{x^3+2x^2+2x+c_2} \qquad (1.2)
\end{aligned}
$$

Here, it is important to notice the $\pm$ that is left in the answer. We will discuss its importance in section 1.3.

1.2 Linear ODEs and Integrating Factors

1.2.1 Linear ODEs

ODEs that are in the form

$$
L\left[y(t)\right] = f(t), \qquad (1.3)
$$

where L is a linear, differential operator, is said to be linear. L can be thought of having the following form:

$$
L_n(y) = \frac{d^n y}{dt^n} + A_i(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + A_{n-1}(t)\frac{dy}{dt} + A_n(t)y \qquad (1.4)
$$

1.2.2 Integration via Integrating Factor

Now let's start to broaden our repertoire of ODE-solving techniques. Let's consider equations in the form

$$
\frac{dy}{dt} + p(t)y(t) = g(t), \qquad (1.5)
$$

where $p(t)$ and $g(t)$ are known functions, and we are solving for $y(t)$. This is the general form of a first-order, linear ODE. These types of equations may be solved by make use of an integrating factor. Let's consider for a moment a function $\mu(t)$, the integrating factor, and let's multiply both sides of the equation with it.

$$
\frac{dy}{dt}\mu(t) + \mu(t)p(t)y(t) = g(t). \qquad (1.6)
$$

Let's see if we can find a μ such that the following form can be reached:

$$
\frac{d}{dt}\left[\mu(t)\cdot y(t)\right] = g(t). \qquad (1.7)
$$

Comparing the terms in both, we can see that

$$
\frac{d\mu}{dt} = \mu(t)p(t). \qquad (1.8)
$$

We can easily solve for $\mu(t)$ now:

$$\begin{aligned}
\frac{d\mu}{\mu} &= p(t)dt \\
\int \frac{d\mu}{\mu} &= \int p(t)dt \\
\ln|\mu(t)| &= \int p(t)dt + \tilde{c} \\
\mu(t) &= c\exp\left\{\int p(t)dt\right\} \qquad (1.9)
\end{aligned}$$

Multiplying both sides of our initial ODE by $\mu(t)$ yields

$$c\exp\left\{\int p(t)dt\right\}\left[\frac{dy}{dt} + p(t)y(t)\right] = c\exp\left\{\int p(t)dt\right\}g(t), \qquad (1.10)$$

so it is clear that we can just drop the constant of integration that came out when solving for $\mu(t)$ because it is common to all terms. Let's examine what we now have:

$$\exp\left\{\int p(t)dt\right\}\frac{dy}{dt} + \exp\left\{\int p(t)dt\right\}p(t)y(t) = \exp\left\{\int p(t)dt\right\}g(t)$$

Because we engineered $\mu(t)$ to allow us to reverse the product rule, we can now express the above equation as

$$\begin{aligned}
\frac{d}{dt}\left[y(t)\exp\left\{\int p(t)dt\right\}\right] &= \exp\left\{\int p(t)dt\right\}g(t) \\
y(t)\exp\left\{\int p(t')dt'\right\} &= \int \exp\left\{\int p(t')dt'\right\}g(t)dt + c.
\end{aligned}$$

which gives us the final form

$$y(t) = \exp\left\{-\int p(t')dt'\right\}\left[\int \exp\left\{\int p(t')dt'\right\}g(t)dt + c\right] \qquad (1.11)$$

This result is the general result for any ODE in the form specified in equation (1.5). Let's look at an example:

$$\begin{aligned}
\frac{dy}{dt} + \frac{2}{t}y(t) &= e^t \\
\frac{dy}{dt}\mu(t) + \frac{2}{t}\mu(t)y(t) &= \mu(t)e^t \qquad (1.12)
\end{aligned}$$

We define $\frac{d\mu}{dt} = \frac{2}{t}\mu(t)$. Let's solve for $\mu(t)$:

$$\begin{aligned}\frac{d\mu}{\mu} &= \frac{2}{t}dt \\ \ln|\mu| &= 2\ln|t| + \tilde{c} \\ \mu(t) &= ct^2 \\ \mu(t) &= t^2\end{aligned}$$

Now that we have $\mu(t)$, let's solve the ODE.

$$\begin{aligned}\frac{dy}{dt}\mu(t) + \frac{d\mu}{dt}y(t) &= \mu(t)e^t \\ \frac{d}{dt}\left[y\mu(t)\right] = \mu(t)e^t \; [y\mu(t)] &= \int e^t t^2 dt \\ y(t) &= t^{-2}\int e^t t^2 dt\end{aligned}$$

Let's perform this integral by way of tabular Integration by Parts.

$+$	t^2	e^t	We multiply across and down
$-$	$2t$	e^t	until we know all the next
$+$	2	e^t	terms will be 0.
$-$	0	e^t	

This gives the result

$$\int e^t t^2 dt = e^t\left[t^2 - 2t + 2\right] + c,$$

which gives us $y(t)$:

$$y(t) = t^{-2}e^t\left[t^2 - 2t + 2\right] + ct^{-2} \tag{1.13}$$

Although you should practice going through this procedure so you can solve these kinds of linear first-order ODEs, because we solved for the general solution earlier, we didn't need to. Plugging in $p(t) = \frac{2}{t}$ and $g(t) = e^t$ into equation (1.11), you can check that we get the same answer.

1.3 Initial-Value Problems

All the solutions we've come up with so far have had a constant of integration left in the answer. We know from calculus that we can get rid of this if we have a known point that the function passes through. When solving for ODEs with a given initial value, the problem are referred to as an initial-value problem. Solutions to initial-value problems are always unique, as we will later discuss. Let's look again at the examples in section 1.1, equations (1.1) and (1.2). Let's impose the initial value of $y(0) = y_0$ and see what we get.

Equation (1.1):

$$y(0) = y_0 = c \exp\{-\alpha 0\} \implies c = y_0$$
$$y(t) = y_0 \exp\{\alpha t\}$$

Equation (1.2):

$$y(0) = y_0 = 1 \pm \sqrt{c_2} \implies c_2 = (y_0 - 1)^2$$
$$y(t) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + (y_0 - 1)^2}$$

Hold on. This plus or minus is still there. This can't be right, because we know that solutions to IVPs must be unique. To really solve this IVP, we need additional qualification on y_0 to know which sign to choose. When we equated $y_0 = 1 \pm \sqrt{c_2}$, clearly if $y_0 > 1$, we choose the plus sign, and if $y_0 < 1$, we choose the minus sign. If $y_0 = 1$, we unfortunately still don't have a solution. If we look back at the ODE we started with, we can see that $\frac{dy}{dx}$ is undefined at $y = 1$, so it makes sense that we don't get a solution for $y_0 = 1$.

1.4 Homogeneous Equations

Sometimes we can solve equations by performing the following substitution:

$$v(x) = \frac{y(x)}{x} \tag{1.14}$$

This can sometimes lead us to a separable equation. Let's look at an example:

$$\frac{dy}{dx} = \frac{x^3 + xy + \frac{1}{x}y^3}{x^2 + y^2} = \frac{x + \frac{y}{x} + \frac{y^3}{x^3}}{1 + \frac{y^2}{x^2}} = \frac{x + v + v^3}{1 + v^2}$$

Now we need to get rid of that $\frac{dy}{dx}$ and replace it with something we can actually work with.

$$v(x) = \frac{y(x)}{x}$$
$$y(x) = xv(x)$$
$$\frac{dy}{dx} = v(x) + x\frac{dv}{dx}$$

So now we have something to replace $\frac{dy}{dx}$ with.

$$v(x) + x\frac{dv}{dx} = \frac{x + v + v^3}{1 + v^2}$$
$$x\frac{dv}{dx} = \frac{x + v + v^3}{1 + v^2} - \frac{v + v^3}{1 + v^2}$$
$$x\frac{dv}{dx} = \frac{x}{1 + v^2}$$
$$\frac{dv}{dx} = \frac{1}{1 + v^2}$$

Now we have a separable equation. Let's solve:

$$dv(1+v^2) = dx$$
$$v + \frac{v^3}{3} = x + c$$

And now we have the implicit solution in v:

$$3v + v^3 - 3x - 3c = 0$$

To get this equation back in terms of y, we just undo our original $v = \frac{y}{x}$ substitution.

$$3\frac{y}{x} + \frac{y^3}{x^3} - 3x - 3c = 0$$
$$y^3 + 3yx^2 - 3x^4 - 3cx^3 = 0$$

We don't have to get this in explicit form, though it is possible.

2 Non-Linear Dynamical Systems

We will start our discussion of non-linear dynamical systems (NDS) by discussing autonomous equations.

2.1 Autonomous Equations

What does autonomous mean in this context?

$$\frac{d}{dt}y(t) = f(y(t), \underline{t})$$

That underlined t represents explicit dependence on t. Equations with no explicit dependence on t are said to be autonomous. Let's go through some examples:

$$\frac{dy}{dt} = ay(t) + by^3(t) \implies \text{autonomous}$$
$$\frac{dy}{dt} = a\sin(y(t)) \implies \text{autonomous}$$
$$\frac{dy}{dt} = at\sin(y(t)) \implies \text{non-autonomous}$$
$$\frac{dy}{dt} = ay(t) + t^3 \implies \text{non-autonomous}$$

Autonomous equations are important because we can perform qualitative analysis without actually solving the equations. Exponential growth

$$\frac{dy}{dt} = \lambda y(t) \implies y(t) = C\exp\{\lambda t\}$$

is an easy example of an autonomous system. What if that growing thing ran out of food? We want $\frac{dy}{dt}$ to be negative at large $y(t)$. So let's redefine $\frac{dy}{dt}$ in this way:

$$\frac{dy}{dt} = [r - ay(t)]\, y(t)$$

where a is much smaller than r. Let's start by cleaning up the notation:

$$\frac{dy}{dt} = f(y(t)) = r\left[1 - \frac{y(t)}{k}\right] y(t) \tag{2.1}$$

where $k = \frac{r}{a}$. This equation (2.1) is known as the logistic equation, and it was first published in 1838. Without solving the equation, we can perform some qualitative analysis to understand what the solutions look like. Even though our equation is solvable (and better yet separable!), we will use it to get a feel for qualitative analysis.

2.2 Qualitative Analysis

Qualitative analysis is very helpful for determining what the solutions to a particularly ODE will look like, even for unsolvable systems. We will explore this using our logistic equation (2.1).

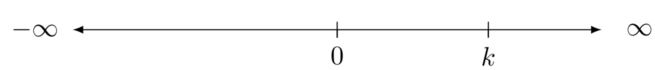
Let's start by finding the simplest possible solutions: where $y(t)$ is a constant. We'll call $y(t)$ that has been set explicitly constant $\bar{y}$. It follows that

$$\frac{d\bar{y}}{dt} = 0 = r\left[1 - \frac{\bar{y}}{k}\right]\bar{y}.$$

This gives the following solutions for $\bar{y}$:

$$\bar{y}_1 = 0$$
$$\bar{y}_2 = k$$

These solutions for $\bar{y}$ are referred to as the equilibrium points, equilibria, steady-state points, or fixed points of $y(t)$. We can now start a "phase portait" for $y(t)$.



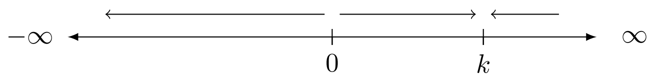
We want to make a statement about the solutions for all initial values of y. Let's go through all the possibilities of $y(0)$.

Let's start with $y(0) = y_0 > k$: Because y_0 is not a steady state, so it must either increase or decrease. If we stick this y_0 into the equation, we can see that $y_0 > k$ makes $\frac{dy}{dt} < 0$.

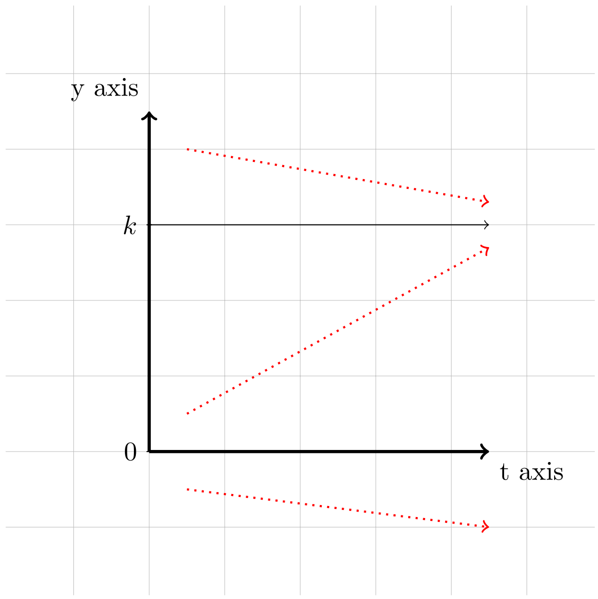
Now let's consider $0 < y_0 < k$: Plugging this into the equation, we can see that $\frac{dy}{dt} > 0$.

There is one more option, although not physical possible ("we cannot have negative rabbits", as professor Dorning put it), it is nice to know just so we can more deeply understand the solutions to the ODE. Considering $y_0 < 0$, we see that $\frac{dy}{dt} < 0$.

Let's put this information on our phase portrait:



Now let's look at a graph:



We still don't know what the curvature of the solutions will be. Finding the curvature is just a matter of differentiating y again.

$$\begin{aligned}
\frac{df}{dt} &= \frac{df}{dy}\frac{dy}{dt} \\
&= \frac{df}{dy} f \\
&= r\left[1 - \frac{2y}{k}\right] \cdot r\left[1 - \frac{y}{k}\right] y
\end{aligned}$$

2.3 Bernoulli Equations

Equations of the following form are known as *Bernoulli Equations*:

$$\frac{dy}{dt} + p(t)y(t) = q(t)y^n(t). \tag{2.2}$$

These equations can be solved by performing the following substitution:

$$v = y^{1-n}$$
$$\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt}$$
$$\frac{1}{1-n}\cdot\frac{dv}{dt} + p(t)v = q(t)$$

3 Exact ODEs

3.1 The Big Picture

Suppose there exists a function

$$\psi(x, y(x)) = \text{const}, \tag{3.1}$$

then

$$\frac{d}{dx}\psi(x, y, (x)) = \frac{d}{dx}\text{const} = 0, \tag{3.2}$$

$$\begin{aligned}\frac{d}{dx}\psi(x, y, (x)) &= \frac{\partial}{\partial x}\psi(x, y(x)) + \frac{\partial}{\partial y}\psi(x, y(x))\frac{dy}{dx}\\ &= \psi_x(x, y(x)) + \psi_y(x, y(x))\frac{dy}{dx}\\ &= M(x, y(x)) + N(x, y(x))\frac{dy}{dx} = 0\end{aligned}$$

So if we have an ODE in the form of

$$M(x, y(x)) + N(x, y(x))\frac{dy}{dx} = 0, \tag{3.3}$$

we can solve it by trying to find a ψ such that $\psi_x = M$ and $\psi_y = N$. These types of equations are referred to as "exact ODEs" because the exact differential $d\psi$ equals 0, as seen here:

$$\frac{d\psi}{dx} = 0 \implies d\psi = 0.$$

3.2 Checking if an ODE is Exact

If we remember back to Clairaut's theorem from multivariable calculus, we know that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\vec{F}(x, y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\vec{F}(x, y),$$

and if we apply that to our discussion of exact ODEs, we can see that

$$\psi_{xy} = \psi_{yx}$$
$$M_y = N_x.$$

So as we can see, M_y *must* equal N_x for an ODE to be exact. You must remember to check if an equation is exact before you attempt to find $\psi(x, y(x))$, otherwise you will be wasting time. Professor Dorning recommended the mnemonic "my" $\rightarrow M_y$ comes first. When $M_y = N_x$, the ODE is *always* exact, meaning there will *always* be a ψ such that $\psi_x = M, \psi_y = N$.

3.3 Solving Exact ODEs

To solve for $\psi(x, y(x))$, we need to integrate M (dx) and N (dy) and examine the results. We can best see this by doing an example:

ODE: $2x + y^2(x) + (2xy(x) + 2y(x))\dfrac{dy}{dx} = 0$; $x \neq 0$; BVC: $y(x = 1) = -2$

First we check to see if the ODE is exact:

$$M_y = 2y(x)$$
$$N_x = 2y(x),$$

so the ODE is exact, and we can solve for ψ. Let's start with M. Because $\frac{\partial}{\partial x}\psi(x, y(x)) = M(x, y(x))$:

$$\begin{aligned}\psi(x, y(x)) &= \int M(x, y(x))dx \\ &= \int (2x + y^2)dx \\ &= x^2 + xy^2 + c(y)\end{aligned}$$

When integrating with respect to x, the result will have a "constant" that is an arbitrary function of y. This function of y will be determined when integrating N. Now let's integrate N.

$$\begin{aligned}\psi(x, y(x)) &= \int N(x, y(x))dy \\ &= \int (2xy + 2y)dy \\ &= xy^2 + y^2 + c(x)\end{aligned}$$

Comparing these, we can see that

$$\psi(x, y(x)) = xy^2 + x^2 + y^2 + \tilde{c}$$

Because we know that $\psi(x, y(x)) = c$, we can just move $\tilde{c}$ over with the c. Therefore, we can just say

$$\psi(x, y(x)) = xy^2 + x^2 + y^2 = c$$

This gives us the implicit solution to our exact ODE. This type of solution can be converted to explicit form

$$\begin{aligned} xy^2 + x^2 + y^2 &= c \\ y^2 &= \frac{1}{x+1}\left(c - x^2\right) \\ y &= \pm\left[\frac{1}{x+1}\left(c - x^2\right)\right]^{\frac{1}{2}} \end{aligned}$$

Applying the boundary value condition to the implicit form (because it's easier):

$$\begin{aligned} 1 \cdot (-2)^2 + 1^2 + (-2)^2 &= c \\ c &= 9 \\ y &= \pm\left[\frac{1}{x+1}\left(9 - x^2\right)\right]^{\frac{1}{2}} \end{aligned}$$

We drop the $\pm$ by re-applying the BVC.

$$\begin{aligned} -2 &= \pm\left[\frac{1}{1+1}\left(9 - (-1)^2\right)\right]^{\frac{1}{2}} \\ -2 &= -\left[\frac{1}{1+1}\left(9 - (-1)^2\right)\right]^{\frac{1}{2}} \\ y &= -\left[\frac{1}{x+1}\left(9 - x^2\right)\right]^{\frac{1}{2}} \end{aligned}$$

It is important to notice that if the BVC were given at $x = -1$, we wouldn't have a solvable system (we wouldn't be able to generate a unique solution). We would be able to solve the ODE, but not the BVP.

We can solve for ψ, instead of by integrating twice and then comparing the results, by integrating, differentiating, and then integrating again. This method requires a little more thought and may seem less intuitive. If we take our last ODE:

$$\begin{aligned} \psi(x, y(x)) &= \int M(x, y(x))dx \\ \psi(x, y(x)) &= x^2 + xy^2 + c(y) \\ \frac{\partial \psi}{\partial y} &= 2xy + c'(y) \text{ (from differentiating)} \\ &= N(x, y(x)) \\ &= 2xy + 2y \text{ (known from the problem)} \end{aligned}$$

In this case, $c'(y) = 2y$, so $c(y) = y^2 + \tilde{c}$, the same result we got when we just integrated M and N and compared the result. Putting this back into our equation for ψ, we get

$$\psi(x, y(x)) = x^2 + xy^2 + y^2 = c,$$

exactly the same result as before. This process of solving for ψ might save time, but it also might not.

3.4 Solving Inexact ODEs by Making Them Exact

Let's say we had an ODE that wasn't exact. Can we make it exact? Short answer: yes. Some ODEs can get very complicated when trying to solve them this way, but others can be solved by hand. We can make these inexact equations exact by multiplying through by another type of integrating factor $\mu(x, y(x))$ and forcing the result to satisfy the requirements of an exact equation.

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

There are some conditions such that $\mu(x, y)$, although making the equation exact, makes the equation very hard to solve (i.e. it turns the problem into a partial differential equation (PDE)). However, if μ depends either only one x or y, it makes solving the ODE easier.

Let's first consider $\mu(x)$, where x is the only variable μ depends on:

$$\begin{aligned}
\mu M + \mu N \frac{dy}{dx} &= 0 \\
\frac{\partial}{\partial y}[\mu M] &= \frac{\partial}{\partial x}[\mu N] \\
M\frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} &= N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}
\end{aligned}$$

Because we know that μ has no dependence on y, $\frac{\partial \mu}{\partial y} = 0$. This gives us the following:

$$\begin{aligned}
\mu\frac{\partial M}{\partial y} &= \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x} \\
\frac{d\mu}{dx} &= \mu\frac{M_y - N_x}{N} \qquad (3.4)
\end{aligned}$$

This can be evaluated if $\frac{M_y - N_x}{N}$ is a function only of x, and if it is, we know that μ depends only on x, and we can easily find that $\mu(x)$ to make our original ODE exact.

Now let's consider $\mu(y)$, where y is now the only variable μ depends on. The analysis is very similar.

$$M\frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

Because we know that μ has no dependence on x, $\frac{\partial \mu}{\partial x} = 0$. This gives us the following:

$$\begin{aligned} M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} &= \mu\frac{\partial N}{\partial x} \\ \frac{d\mu}{dy} &= \mu\frac{N_x - M_y}{M} \end{aligned} \tag{3.5}$$

Similar to our last analysis, if $\frac{N_x - M_y}{M}$ is a function of just y, we can solve for $\mu(y)$, were μ is a function of only y. Multiplying our original ODE through by $\mu(y)$ should give us an exact ODE.

Let's do an example:

$$(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$$

Let's first check if this ODE is exact:

$$\begin{aligned} M_y &= 3x + 2y \\ N_x &= 2x + y \end{aligned} \quad \Longrightarrow \quad \text{not exact}$$

Now we need to check if we can find a μ with a dependence on only one variable. Checking for an x-dependent μ first using equation (3.4):

$$\begin{aligned} \frac{M_y - N_x}{N} &= \frac{3x + 2y - (2x + y)}{x^2 + xy} \\ &= \frac{x + y}{x^2 + xy} \\ &= \frac{x + y}{x(x + y))} \\ &= \frac{1}{x} \end{aligned}$$

So now we know there exists a μ such that it only depends on x. If $\frac{M_y - N_x}{N}$ turned out to depend both on y as well (or even on y solely), we'd need to check to see if equation (3.5) was a function of just y to check for y dependence of μ instead. We got lucky this time. Now let's solve for $\mu(x)$.

$$\begin{aligned} \frac{d\mu}{dx} &= \mu\left(\frac{M_y - N_x}{N}\right) \\ \frac{d\mu}{dx} &= \mu\frac{1}{x} \\ \int \frac{d\mu}{\mu} &= \int \frac{dx}{x} \\ \ln|\mu| &= \ln|x| + c \\ |\mu| &= |x|e^c \\ \mu &= \pm xe^c \\ \mu &= cx \end{aligned}$$

We can set $c = 1$ because the integrating factor is multiplied into the whole equation, so that constant factor can just be removed. This is the same removal we did with the other integrating factors we talked about in the very beginning of these notes. Now let's multiply our equation by $\mu(x) = x$:

$$\begin{aligned} x[(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} &= 0] \\ (3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} &= 0 \\ \tilde{M}(x,y) + \tilde{N}(x,y)\frac{dy}{dx} &= 0 \end{aligned}$$

Let's now check if this new ODE is exact:

$$\begin{matrix} \tilde{M}_y = 3x^2 + 2xy \\ \tilde{N}_x = 3x^2 + 2xy \end{matrix} \implies \text{exact!}$$

Mission accomplished! We turned an inexact ODE into an exact one. Let's solve the exact ODE just for practice:

$$\begin{aligned} \psi(x,y) &= \int \tilde{M}(x,y)dx \\ &= x^3y + \frac{1}{2}x^2y^2 + c(y) \\ &= \int \tilde{N}(x,y)dy \\ &= x^3y + \frac{1}{2}x^2y^2 + c(x) \\ &= x^3y + \frac{1}{2}x^2y^2 = c \end{aligned}$$

Because this equation is quadratic in y, you should be able to get this in explicit form, but I won't do that here.

3.5 Second-Order ODEs as First-Order ODEs

The following types of ODEs can be solved using the techniques we have learned for first-order ODEs.

3.5.1 Riccati Equations

The Riccati Equation is the following:

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2 \tag{3.6}$$

Let's suppose an ODE in this form has a known solution $y_1(t)$. It was found that a more general solution can be found if we perform the substitution

$$y = y_1(t) + \frac{1}{v(t)}.$$

It can be shown that $v(t)$ satisfies the first-order ODE *linear* equation

$$\frac{dv}{dt} = -(q_2 + 2q_3 y_1)v - q_3$$

3.5.2 Equations Missing the Dependent Variable

Let's consider second-order ODEs in the following form:

$$y'' = f(t, y') = f(t, v)$$

We perform the following substitution:

$$y' = \frac{dy}{dt} = v(t)$$

After performing the substitution, we can solve for v using the ODE as if it were a first-order ODE, and then we integrate v with respect to t to find y.

3.5.3 Equations Missing the Independent Variable

Let's consider second-order ODEs in the following form:

$$y'' = f(y, y') = f(y, v)$$

We perform the following substitution:

$$y' = \frac{dy}{dt} = v(t) \implies \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

Let's do a quick example:

$$\begin{aligned} yy'' + (y')^2 &= 0 \\ yv' + v^2 &= 0 \end{aligned}$$

Now let's solve for v:

$$\begin{aligned} \frac{dv}{dt} &= -\frac{v^2}{y} \\ v\frac{dv}{dy} &= -\frac{v^2}{y} \\ \frac{dv}{v} &= \frac{-1}{y}dy \\ \ln|v| &= -\ln|y| + c \\ v &= c_1 y^{-1} \end{aligned}$$

Substituting $\frac{dy}{dt}$ back in for v, we can solve for y:

$$\begin{aligned}\frac{dy}{dt} &= c_1 y^{-1} \\ \int y dy &= \int c_1 dt \\ y^2 &= 2(c_1 t + c_2) \\ y^2 &= \tilde{c}_1 t + \tilde{c}_2\end{aligned}$$

This is the general stricture of a problem like this.

4 Homogeneous Second-Order ODEs

Solving second-order differential equations has so many uses through many fields of study, so these equations are important to know how to solve. Second-order ODEs have the following form:

$$\frac{d^2y}{dt^2} = f\left(t, y(t), \frac{dy}{dt}(t)\right) \tag{4.1}$$

If $f(t, y(t), \frac{dy}{dt}(t))$ is linear in y and $\frac{dy}{dt}$, then we say the second-order ODE is linear, and we can write it in the following form:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y(t) = g(t) \tag{4.2}$$

This form is analogous to the form we described for first-order ODEs. This form is also called the standard form for second-order ODEs, as the coefficient of the highest-order derivative is 1. It can always be made 1, because if it were a function of t, we could just divide the whole equation by that function of t to return to standard form.

4.1 Boundary Conditions

Let's examine the following second-order ODE:

$$\begin{aligned}\frac{d^2y}{dt^2}(t) &= 0 \\ \frac{dy}{dt}(t) &= c_1 \\ y(t) = c_1 t + c_2 &\end{aligned}$$

Because, for second-order ODEs, at some point we must integrate twice in some form or another, we will always have *two* arbitrary constants to deal with. Often times these two constants are given in the following way:

$$\begin{aligned}y(0) &= y_0 \\ y'(0) &= y'_0,\end{aligned}$$

and you'd solve for the cs in the only logical way:

$$\begin{aligned} y'(0) = c_1 = y'_0 \\ y(0) = c_1(0) + c_2 = c_2 = y_0 \end{aligned} \implies y(t) = y'_0 t + y_0$$

The initial values can also be given with just the y function (e.g. $y(0) = y_1$ and $y(2) = y_2$), or just with the first derivative of y (e.g. $y'(0) = y'_1$ and $y'(2) = y'_2$). They are both equally solvable.

4.2 The Solutions to Basic Second-Order ODEs

Let's consider the following ODE:

$$y''(t) - y(t) = 0 \tag{4.3}$$

It was found out that the solutions to these ODEs are invariant in t, or invariant under time translation. One of the functions we know that exhibits that property is the exponential:

$$e^{t+\tau} = \tilde{\tau} e^t$$

Because of this, we know that solutions to ODE (4.3) will be in the form of an exponential:

$$\begin{aligned} y(t) &= Ce^{rt} \\ y''(t) &= r^2 Ce^{rt} \end{aligned}$$

Substituting this back into our ODE, we got the following:

$$\begin{aligned} y''(t) - y(t) &= 0 \\ Ce^{rt}(r^2 - 1) &= 0 \\ \\ C = 0 \text{ or } r &= \pm 1 \end{aligned}$$

$C = 0$ represents the trivial solution (putting it back into our equation quickly yields $0 = 0$), so we don't care about it. We do care about $r = \pm 1$. So the solutions we care about are in the following forms:

$$y_1(t) = C_1 e^{-t} \tag{4.4}$$

$$y_2(t) = C_2 e^{t} \tag{4.5}$$

But hang on, that means for any particular solution, there is only one arbitrary constant. There should be two because there are always two when solving second-order differential equations. We'll come back to this problem after our discussion of the superposition theorem of solutions to ODEs.

4.3 Superposition Theorem

Looking back at our solutions to ODE (4.3), we now know we can combine equations (4.4) and (4.4) to come up with the following general solution:

$$y(t) = C_1 e^{-t} + C_2 e^t. \tag{4.6}$$

You can check to see that this equation satisfies our original ODE. This property comes directly from the linearity of the differential operator, d.

4.4 Second-Order Homogeneous ODEs of Constant Coefficients

Let's consider ODEs with constant coefficients, such that the ODEs we are care about in the following form:

$$ay'' + by' + cy = 0. \tag{4.7}$$

From our previous discussion, we know that the solutions are the following form:

$$y(t) = Ce^{rt}$$

Plugging this into our ODE yields

$$ar^2Ce^{rt} + brCe^{rt} + cCe^{rt} = 0$$

Because we know that e^{rt} can never be zero, we can just factor it out, yielding

$$C(ar^2 + br + c) = 0$$

This is what we call the *characteristic equation.* As with before, $C = 0$ is the trivial solution, so we really just care about the r terms, the *roots of the characteristic equation.* Using the quadratic formula, we know that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now let's look at the discriminant (the argument of the square root, $b^2 - 4ac$) to determine the format of the solutions: Here are the three possibilities for the discriminant:

1. $b^2 - 4ac > 0 \implies$ two real, distinct solutions.
2. $b^2 - 4ac = 0 \implies$ one real, repeated solution.
3. $b^2 - 4ac < 0 \implies$ a complex conjugate pair of solutions.

4.4.1 Distinct, Real Solutions

For now we will consider case 1: the discriminant is positive. The general solution to the ODE will be

$$y(t) = C_1 \exp\left\{t \cdot \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right\} + C_2 \exp\left\{t \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

Example Let's consider the following ODE:

$$y'' + 2y' - 3y = 0$$

Because we know the solutions will be in the form $y(t) = Ce^{rt}$, we put this into the ODE to get the following:

$$\begin{aligned} Ce^{rt}(r^2 + 2r - 3) &= 0 \\ (r^2 + 2r - 3) &= 0 \\ (r + 3)(r - 1) &= 0 \implies r_1 = -3, r_2 = 1 \end{aligned}$$

We can drop the exponential part as it is never 0, and we don't care of C is 0, because it would just give us the trivial solution. This gives us the following solution to the ODE:

$$y(t) = c_1 e^{-3t} + c_2 e^t$$

4.4.2 Complex Conjugate Roots

If the discriminant is less than 0, we can rewrite the roots as the following:

$$\begin{aligned} r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac)} \\ r &= \frac{-b}{2a} \pm \frac{1}{2a}\sqrt{-(4ac - b^2)} \\ r &= \frac{-b}{2a} \pm \frac{i}{2a}\sqrt{4ac - b^2} \\ r &= \lambda \pm i\mu \end{aligned}$$

We're going to refer to the real part of the complex conjugate roots as r and the complex part as μ. The sign of μ doesn't matter because the roots are complex conjugates, so normally we just take μ to be positive. Let's plug in this result into the equation for the general result for second-order ODEs:

$$\begin{aligned} y(t) &= A_1 e^{r_1 t} + A_2 e^{r_2 t} \\ &= A_1 \exp\{(\lambda + i\mu)t\} + A_2 \exp\{(\lambda - i\mu)t\} \\ &= A_1 e^{\lambda t} e^{i\mu t} + A_2 e^{\lambda t} e^{i\mu t} \end{aligned}$$

Here we use Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \tag{4.8}$$

and then we use the properties of cos (even) and sin (odd) allow us to pull out the negative signs.

$$\begin{aligned} y(t) &= A_1 e^{\lambda t}\left[\cos(\mu t) + i\sin(\mu t)\right] + A_2 e^{\lambda t}\left[\cos(-\mu t) + i\sin(-\mu t)\right] \\ &= A_1 e^{\lambda t}\left[\cos(\mu t) + i\sin(\mu t)\right] + A_2 e^{\lambda t}\left[\cos(\mu t) - i\sin(\mu t)\right] \\ &= e^{\lambda t}\left[(A_1 + A_2)\cos(\mu t) + i(A_1 - A_2)\sin(\mu t)\right] \end{aligned}$$

Now we can start replacing the A_1 and A_2 terms with constants. Keep in mind these arbitrary constants can be complex too, so is are included.

$$y(t) = e^{\lambda t}\left[c_1 \cos(\mu t) + c_2 \sin(\mu t)\right]$$

Here c_1 and c_2 are real if a, b, c, and initial-value conditions are all real. Now we can decompose $y(t)$ into our fundamental solutions:

$$\begin{aligned} y_1(t) &= e^{\lambda t}\cos(\mu t) \\ y_2(t) &= e^{\lambda t}\sin(\mu t) \end{aligned}$$

Example Let's consider the following ODE:

$$y'' + 2y' + 4y = 0$$

This gives us the following roots:

$$\begin{aligned} r^2 + 2r + 4 = 0 \\ r &= \frac{-2 \pm \sqrt{4 - 16}}{2} \\ &= -1 \pm i\sqrt{3} \end{aligned}$$

So we can set $\mu = \sqrt{3}$, and $\lambda = -1$. This gives us the following solution:

$$y(t) = e^{-t}\cos(3t) + e^{-t}\sin(3t)$$

4.4.3 Repeated Real Roots

Let's consider the third and final case for the discriminant, the case where it equals 0.

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ r &= \frac{-b}{2a} = r_1 = r_2 \end{aligned}$$

This gives us the following:

$$\begin{aligned} y_1(t) &= c_1 \exp\left\{\frac{-b}{2a}t\right\} \\ y_2(t) &= c_2 \exp\left\{\frac{-b}{2a}t\right\} \\ y(t) &= (c_1 + c_2) \exp\left\{\frac{-b}{2a}t\right\} \\ &= c_3 \exp\left\{\frac{-b}{2a}t\right\} \end{aligned}$$

So we only have one solution! How are we supposed to get another solution? This is a very important question, and what follows is a very important procedure, as Professor Dorning likes to say. Given

$$y_1(t) = c_1 e^{r_1 t},$$

we need to look for solutions in the following form:

$$y_2(t) = \tilde{c}_2 e^{r_1 t}.$$

Let $\tilde{c}_2$ be a function of t, $v(t)$:

$$y_2(t) = v(t) e^{r_1 t}.$$

Throwing this back into the ODE that gave us the first solution will give us a first-order ODE to solve for $v(t)$, and that will yield a second solution. Let's do it:

$$a y_2'' + b y_2' + c y_2 = 0 \tag{4.9}$$

Let's calculate y'' and y':

$$\begin{aligned} y_2(t) &= v(t) e^{r_1 t} \\ y_2'(t) &= (v'(t) + r_1 v) e^{r_1 t} \\ y_2''(t) &= (v''(t) + 2r_1 v' + r_1^2 v) e^{r_1 t} \end{aligned}$$

Plugging this into equation (4.9), we get

$$e^{r_1 t}\left\{[av'' + a2r_1 v' + ar_1^2 v] + [bv' + br_1 v] + cv\right\} = 0$$

Using the fact that $r_1 = \frac{-b}{2a}$ and $b^2 = 4ac$, we get the following relation:

$$\begin{aligned} e^{r_1 t}\left\{\left[av'' - br_1 v' + \frac{b^2}{4a} r_1^2 v\right] + \left[bv' - \frac{b^2}{2a} v\right] + \frac{b^2}{4a} v\right\} &= 0 \\ e^{r_1 t}\left\{av''\right\} &= 0 \\ av'' &= 0 \\ v'' &= 0 \end{aligned}$$

From this we know that

$$v(t) = d_1 t + d_2$$

Therefore, we can conclude the following:

$$\begin{aligned}
y_2(t) &= (d_1 t + d_2)e^{r_1 t} \\
y(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1 e^{r_1 t} + (d_1 t + d_2)e^{r_1 t} \\
&= (c_1 + d_2)e^{r_1 t} + d_1 t e^{r_1 t} \\
&= \tilde{c}_1 e^{r_1 t} + \tilde{c}_2 t e^{r_1 t}
\end{aligned}$$

This gives us the final fundamental solutions

$$\begin{aligned}
y_1(t) &= e^{r_1 t} \\
y_2(t) &= t e^{r_1 t}
\end{aligned} \tag{4.10}$$

We don't have to derive this form ever again, and we can just use this in problems without rederiving it. Are we done? No. We have to check the Wronskian:

$$\begin{aligned}
W(e^{r_1 t}, t e^{r_1 t}) &= \begin{vmatrix} e^{r_1 t} & t e^{r_1 t} \\ r_1 e^{r_1 t} & (1 + r_1 t)e^{r_1 t} \end{vmatrix} \\
&= e^{r_1 t}\left(e^{r_1 t} + t r_1 e^{r_1 t} - t r_1 e^{r_1 t}\right) \\
&= e^{2 r_1 t} \neq 0 \ \forall \ t \in \mathbb{R}
\end{aligned}$$

You can read more about the Wronksian in section 4.5.1. So yes, these are indeed the fundamental solutions to the ODE.

Example Let's consider the following ODE:

$$y'' - 2y' + y = 0$$

Now we get the roots from the characteristic equation by checking for equations of the form $y = Ce^{rt}$:

$$\begin{aligned}
C(r^2 - 2r + 1) &= 0 \\
(r^2 - 2r + 1) &= 0 \\
(r - 1)^2 &= 0 \implies r = 1
\end{aligned}$$

Using the form we derived previously (4.10), we get the following solution:

$$y(t) = c_1 e^t + c_2 t e^t$$

4.5 Second-Order Homogeneous ODEs of Variable Coefficients

4.5.1 The Wronskian

When discussing homogeneous equations, how do we know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

represents all solutions? Can c_1 and c_2 be chosen such that all possible initial value conditions can be satisfied? Let's examine. Consider the following initial conditions:

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \\ \therefore c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ \text{and } c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

Solving for c_1:

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} \tag{4.11}$$

Solving for c_2:

$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} \tag{4.12}$$

Clearly for c_1 and c_2 to be defined, the following must be true:

$$y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$$

We can write this as a determinant:

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$$

This determinant is what we call the *Wronskian.*

$$W(y_1(t_0), y_2(t_0)) \neq 0$$

The Wronskian is used to determine the uniqueness (linear independence) of solutions. If the Wronksian for two solutions is non-0, then the solutions represent distinct (linear independent) solutions. If the Wronskian is 0, then you haven't found all the solutions yet. It is important that the Wronskian depends on initial conditions, if the Wronskian is 0 anywhere (defined by initial conditions), then it is 0 everywhere. Similarly, if the Wronskian is ever non-0, it is always non-0. If the Wronksian of any two solutions is non-0, there is a choice of c_1 and c_2 such that the following solution arises:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

This was proved when solving for the valid c_1 and c_2 values.

General Solution Theorem This theorem sets out to answer the question: "do we have all the solutions?" The theorem is as follows:

Let $y_1(t)$ and $y_2(t)$ be two solutions to the homogeneous ODE

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \tag{4.13}$$

and let $W(y_1(t), y_2(t)) \neq 0$ at a point $t = t_0$. Then, the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{4.14}$$

includes every solution to the ODE.

Let's prove this.

1. Let $\phi(t)$ be any solution to equation (4.13).

2. Let t_0 be a point where $W(y_1(t), y_2(t)) \neq 0$.

3. Evaluate $\phi(t)$ and $\phi'(t)$ at t_0:

$$y_0 = \phi(t_0); \quad y_0' = \phi'(t_0)$$

4. Consider the IVP we have just constructed:

$$y'' + p(t)y' + q(t)y = 0; y(t_0) = y_0; y'(t_0) = y_0'$$

   4.1. Clearly $\phi(t)$ is a solution

   4.2. Since $W(y_1(t), y_2(t)) \neq 0$, c_1 and c_2 can be chosen such that $c_1 y_1(t) + c_2 y_2(t)$ is a solution to the IVP.

   4.3. Because both of the solutions $\phi(t)$ and $c_1 y_1(t) + c_2 y_2(t)$ solve the same IVP and all solutions to IVPs are unique,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t).$$

   Thus, for any $\phi(t)$, it can be represented by our fundamental solutions $y_1(t)$ and $y_2(t)$.

Abel's Theorem Let $y_1(t)$ and $y_2(t)$ be solutions to the following ODE:

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous on the open interval $I = (\alpha, \beta)$. Then

$$W(y_1(t), y_2(t)) = C \exp\left\{-\int p(t)dt\right\}. \tag{4.15}$$

This also implies that, if W is non-0 for any t, then it is non-0 for all t.

Proof To prove this we start by taking advantage of the fact that y_1 and y_2 are solutions:

$$y_1'' + y_1'p(t) + y_1q(t) = 0$$
$$y_2'' + y_2'p(t) + y_2q(t) = 0$$

Now, we multiply the top equation by $-y_2$ and the bottom equation by y_1 and then add the two:

$$(y_1y_2'' - y_2y_1'') + (y_1y_2' - y_2y_1')p(t) = 0$$

Next, we let $W(t) = W(y_1(t), y_2(t)) = y_1y_2' - y_1' - y_2$:

$$(y_1y_2'' - y_2y_1'') + W(t)p(t) = 0$$

Now we observe that $W'(t) = y_1y_2'' - y_1''y_2$, giving us the following:

$$W'(t) + W(t)p(t) = 0$$

We can immediately solve this because this is a separable, first-order ODE, yielding equation (4.15).

4.5.2 Existence, Uniqueness, and Differentiability

Let's consider the ODE:

$$y'' + p(t)y' + q(t)t = g(t)$$

Notice that this equation is *not* homogeneous. We will not prove the following: if $p(t), q(t)$, and $g(t)$ are continuous on an open interval $I = (\alpha, \beta)$ containing the point t_0, then the IVP has a solution $y = \phi(t)$ (existence). The IVP also has a unique solution, and the solution is defined everywhere on I, where p, q, and g are continuous and which contains t_0, and the solution is at least twice differentiable[1].

4.5.3 Euler Equations

Euler equations are differential equations where the degree of every derivative is multiplied by that degree of the independent variable and a constant. Generally

$$y^{(n)}t^nc_n + \cdots + y^{(1)}t^1c_1 + y^{(0)}t^0c_0 = 0$$

$$\sum_{k=0}^{n} y^{(k)}t^kc_k = 0$$

Notice here $y^{(n)}$ represents the n^{th} derivative of y. A second-order Euler equation would look like the following:

$$t^2y'' + t\alpha y' + \beta y = 0 \tag{4.16}$$

[1]This is sometimes denoted as C^2. This means that f, f', and f'' are all continuous on I.

These are important because, under the following substitution, these equations turn into ODEs of constant coefficients.

$$x = \ln t \implies \frac{dx}{dt} = \frac{1}{t}$$

It can be shown quite easily that equation (4.16) can be rewritten in the following way:

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$$

Once the equations have been put into this form, y can be solved in terms of x, and then you just need to replace every x with $\ln t$. This is very powerful, and the same idea shows up many times in Differential Equations.

4.5.4 Method of Reduction of Order

Let's start with the an ODE of variable coefficients:

$$y'' + p(t)y' + q(t)y = 0 \tag{4.17}$$

If we know one solution $y_1(t)$, we can find the other solution using $\tilde{y}_2(t) = v(t)y_1(t)$. Let's start by differentiating $\tilde{y}_2(t)$:

$$\begin{aligned} \tilde{y}_2' &= v'y_1 + vy_1' \\ \tilde{y}_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Substituting this back into equation (4.17), we get

$$\begin{aligned} (v''y_1 + 2v'y_1' + vy_1'') + p(t)(v'y_1 + vy_1') + q(t)v(t)y_1 &= 0 \\ v''y_1'' + (2y_1' + p(t)y_1)v' + \underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{\text{Goes to 0}}v &= 0 \end{aligned}$$

Because y_1 is a solution to equation (4.17), we can just remove the labelled term. This leaves us with the final result

$$\begin{aligned} v''y_1'' + (2y_1' + p(t)y_1)v' &= 0 \\ u'y_1'' + (2y_1' + p(t)y_1)u &= 0 \end{aligned}$$

where $u(t) = v'(t)$. This is now a first-order ODE, which is why we call it the Method of Reduction of Order. Let's continue:

$$\begin{aligned} v'' + \underbrace{\frac{2y_1' + p(t)y_1}{y_1}}_{\tilde{p}(t)}v' &= 0 \\ v'' + \tilde{p}(t)v' &= 0 \end{aligned}$$

Here we make the substitution $u = v'$.

$$\begin{aligned}
u' + \tilde{p}(t)u &= 0 \\
u &= c \exp\left\{-\int \tilde{p}(t)dt\right\} \\
v &= c_1'' \int \exp\left\{-\int \tilde{p}(t)dt\right\} dt + c_2''
\end{aligned}$$

Now we put v back into the equation for $y(t)$:

$$\begin{aligned}
y(t) &= c_1' y_1(t) + c_2' \tilde{y}_2(t) \\
&= c_1' y_1(t) + c_2' y_1 \left[c_1'' \int \exp\left\{-\int \tilde{p}(t)dt\right\} dt + c_2''\right] \\
&= c_1' y_1(t) + c_2 y_1 \int \exp\left\{-\int \tilde{p}(t)dt\right\} dt \\
&= c_1' y_1(t) + c_2 y_1 \int \exp\left\{-\int \frac{2y_1'}{y_1} p(t)dt\right\} dt
\end{aligned}$$

5 Non-Homogeneous Second-Order ODEs

5.1 Method of Undetermined Coefficients

Consider the linear operator L defined in the following way:

$$L[y](t) = g(t) \tag{5.1}$$

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t) \tag{5.2}$$

Let's say we have a function $y_c(t)$ that satisifies the following relation:

$$L[y_c](t) = 0$$

If we define a solution to equation (5.1) $y(t)$ in the following way

$$y(t) = y_c(t) + y_p(t),$$

we can perform the following analysis:

$$\begin{aligned}
L[y](t) &= g(t) \\
L[y_c + y_p](t) &= g(t) \\
L[y_c](t) + L[y_p](t) &= g(t) \\
L[y_p](t) &= g(t) \qquad (5.3)
\end{aligned}$$

That means $y_c(t)$ corresponds to the solution to the homogeneous form of the non-homogeneous ODE, and it's referred to as the "complementary solution." This also means that $y_p(t)$ corresponds to the solution to the non-homogeneous ODE (although not the general solution), and it's referred to as the "particular solution." So how do we solve for $y_p(t)$?

1. Solve the homogeneous form of equation (5.2).

$$L[y_c] = 0 \rightarrow y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

2. Assume a functional form for $y_p(t)$, based on $g(t)$ and its derivatives[2], with undetermined coefficients, and substitute it into equation (5.3) to determine the coefficients.

3. Apply initial conditions to determine coefficients c_1 and c_2 in

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

after combining $y_p(t)$ and $y_c(t)$.

Because we rely on the use of each term of $g(t)$ and its derivatives, we can only use functions that have a finite number of linearly independent derivatives. That means we can only see $g(t)$s including exponentials, sines and cosines, and polynomial terms.

Simple Example

$$y'' - 3y' - 4y = 3e^{2t} \tag{5.4}$$

First let's solve for the complementary solution $y_c(t)$.

$$\begin{aligned} y_c'' - 3y_c' - 4y_c &= 0 \qquad y_c(t) = Ce^{rt} \\ C(r^2 - 3r - 4) &= 0 \\ (r-4)(r+1) &= 0 \implies r = 4, -1 \end{aligned}$$

This gives us

$$y_c(t) = c_1 e^{4t} - c_2 e^{-t}$$

Now we generate $y_p(t)$ by looking at $g(t)$ and its linearly independent derivatives. Because $g(t)$ is an exponential, there are no linearly independent derivatives of $g(t)$, so we can say that $y_p(t) = Ae^{2t}$. Plugging this back into equation (5.4), we get

$$4Ae^{2t} - 3(2Ae^{2t}) - 4Ae^{2t} = 3e^{2t} \implies A = -\frac{1}{2}$$

This gives the following:

$$\begin{aligned} y_p(t) &= -\frac{1}{2}e^{2t} \\ y(t) &= y_c(t) + y_p(t) \\ &= c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t} \end{aligned}$$

[2]Here we don't discuss what this really means, and Professor Dorning left it unclear in the lecture. This will be made clear with various examples in the following discussion.

More Complicated Example

$$\underbrace{y'' - 3y' - 4y}_{\text{same as before}} = 2\sin(t) \tag{5.5}$$

The complementary solution for this ODE is the same as the previous example:

$$y_c(t) = c_2 e^{4t} - c_2 e^{-t}$$

Here, we get $p(t)$ by looking $\sin(t)$ and its linearly independent derivatives: just $\cos(t)$.

$$p(t) = A\sin(t) + B\cos(t)$$

Put this back into equation (5.5) to solve for A and B. The results are $A = -\frac{5}{17}$ and $B = \frac{3}{17}$.

When should we not use this method though?

1. When coefficents are variable:

$$y'' + p(t)y' + q(t)y = g(t)$$

2. When g(t) has a large number or infinite number of linearly independent derivatives. Example: $g(t) = \frac{1}{t}$. This is why we almost always see exponentials, sines and cosines, and polynomials.

5.1.1 "Tricky Cases"

There are some tricky cases when using the method of undetermined coefficients. In this section, $g_n(t)$ refers to one term of $g(t)$, not all terms. So, for example:

$$ay'' + by' + cy = e^t + \cos(2t) + t\sin(2t) + t^4$$

Here $g_n(t)$ will take on each of the following values: e^t, $\cos(2t)$, $t\sin(2t)$, and t^4, and we will perform the following analysis for *each* of those. Because $\cos(2t)$ is included in the linearly independent derivatives of $t\sin(2t)$, we don't need to perform analysis of that term because it will appear in the complementary solution terms contributed by $t\sin(2t)$. It is very important that we won't change all of $g(t)$.

A1. $ay'' + by' + cy = a_0 + a_1 t + a_2 t^2; \quad c = 0$

Instead of choosing $y_p(t) = A_0 + A_1 t + A_2 t^2$, we choose

$$y_p(t) = t[A_0 + A_1 t + A_2 t^2]$$

Why? First let's solve the homogeneous equation:

$$ay_c'' + by_c' = 0$$
$$au_c' + u_c = 0 \implies u_c(t) = c_0 \exp\left\{\frac{-bt}{a}\right\} y_c \quad = \int u_c dt = -\frac{a}{b} C e^{\frac{-b}{a}t} + c_1$$

Now let's solve the non-homogeneous equation:

$$\begin{aligned} ay'' + by' &= a_0 + a_1 t + a_2 t^2 \\ y'' + \frac{a}{b} y' &= a_0 + a_1 t + a_2 t^2 \end{aligned}$$

We can solve this by integrating factor, yielding

$$\begin{aligned} y(t) &= \underbrace{-\frac{a}{b} C_0 \exp\left\{-\frac{b}{a} t\right\} + c_1}_{y_c} + A_0 t + A_1 t^2 + A_2 t^3 \\ y(t) &= -\frac{a}{b} C_0 \exp\left\{-\frac{b}{a} t\right\} + c_1 + t[A_0 + A_1 t^1 + A_2 t^2] \end{aligned}$$

Also notice the following:

$$\begin{aligned} ay_c'' + by_c' &= 0 \\ ar^2 + br &= 0 \\ r(ar + b) &= 0 \implies r_1 = -\frac{b}{a},\ r_2 = 0 \end{aligned}$$

We have *one* zero root, and *one* term missing.

A2. $ay'' + by' + cy = a_0 + a_1 t + a_2 t^2; \quad b = c = 0$ Instead of choosing $y_p(t) = A_0 + A_1 t + A_2 t^2$, we choose

$$y_p(t) = t^2[A_0 + A_1 t + A_2 t^2]$$

Why? Let's do the same analysis we did before. Solving the homogeneous equation first:

$$\begin{aligned} ay_c'' &= 0 \\ y_c &= c_1 t + c_2, \end{aligned}$$

and then the non-homogeneous form

$$\begin{aligned} ay_p'' &= a_0 t^2 + a_1 t + a_2 \\ y_p &= A_0 t^4 + A_1 t^3 + A_2 t^2 \end{aligned}$$

B1. $ay'' + by' + cy = g_n(t)$ What if $g_n(t)$ or its derivatives are already included in the complementary solution? Let's do an example:

$$\begin{aligned} y'' - 4y &= 3e^{2t} \\ r^2 - 4 &= 0 \implies r = \pm 2 \\ y_c(t) &= c_1 e^{2t} + c_2 e^{-2t} \end{aligned}$$

Given $g_n(t) = 3e^{2t}$, we would like to choose $y_p(t) = Ae^{2t}$, but we can't. Instead we choose

$$y_p(t) = Ate^{2t}$$

Let's consider the following form for $y_p(t)$ and place it into the ODE:

$$y_p(t) = v(t)e^{2t}$$
$$y_p''(t) - 4y_p(t) = 3e^{2t}$$

We can solve for $y_p(t)$, removing all the terms that show up in the complementary solution, yielding:

$$y_p(t) = t\left[Ae^{2t}\right]$$

We multiplied it by t to the *first* power, and e^{2t} corresponds to a root that appears *once*.

B2. $ay'' + by' + cy = g_n(t)$ What if $g_n(t)$ or its derivatives are already included in the complementary solution, and the overlapping solutions were results of repeated roots? Let's do an example:

$$\begin{aligned} y'' + 4y' + 4y &= ae^{-2t} \\ y_c'' + 4y_c' + 4y_c &= 0 \\ r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \implies r = -2 \end{aligned}$$

This gives the complementary solution:

$$y_c(t) = c_1 e^{-2t} + c_2 te^{-2t}$$

The particular solution will be of the following form:

$$y_p(t) = t^2\left[Ae^{-2t}\right]$$

Let's prove this. First we take $y_p(t) = v(t)e^{-2t}$, and then we put it back into the ODE:

$$\frac{d^2}{dt^2}\left[v(t)e^{-2t}\right] + 4\frac{d}{dt}\left[v(t)e^{-2t}\right] + 4v(t)e^{-2t} = ae^{-2t}$$

$$\begin{aligned} &\left[v''(t)e^{-2t} - 2v'(t)e^{-2t} - 2v'(t)e^{-2t} + 4v(t)e^{-2t}\right] \\ &\qquad + 4\left[v'(t)e^{-2t} - 2v(t)e^{-2t}\right] + 4v(t)e^{-2t} = ae^{-2t} \end{aligned}$$

$$\begin{aligned} v''(t)e^{-2t} &= ae^{-2t} \\ v''(t) &= a \\ v'(t) &= at + c_2 \\ v(t) &= \frac{a}{2}t^2 + c_2 t + c_3 \end{aligned}$$

The $\frac{a}{2}$ term goes away because we multiply $v(t)y_p(t)$ by undetermined coefficients, so all coefficients become unknown. This means we can rewrite $y_p(t)$ as

$$v(t) = c_1 t^2 + c_2 t + c_3$$

Because the $c_2 t$ and c_3 terms, when multiplied by e^{-2t}, are already included in the complementary, we know that

$$y_p(t) = c_1 t^2 e^{-2t}$$

exactly what we said would be the case.

What this really means Consider the following general ODE:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = g(t)$$

We are going to go through each term in $g(t)$ and perform this analysis:

1. Differentiate it until you have all linearly independent derivatives.
2. See if any of those of those derivatives (include the zero-th derivative) itself are included in the complementary solution.
3. Multiply all derivatives by t until you have no overlap with the complementary solution.

Let's do an example:

$$y'' - 2y' + 2y = t\cos(t) + \sin(t)$$

Here we can use the quadratic equation to get the roots:

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

This gives us the following complementary solution

$$y_c = c_1 \cos(t) + c_2 \sin(t)$$

Now we must go through each of the terms in $g(t)$. Here we can ignore the $\sin(t)$ because it is already included in the derivatives of $t\cos(t)$. We have the following linearly independent derivatives of $t\cos(t)$ (including itself).

$$t\cos(t),\ t\sin(t),\ \cos(t),\ \sin(t)$$

But we have overlap with the complementary solution. We need to multiply all these terms by t, and after we do we can see that the overlap is gone. This gives us the following form $y_p(t)$:

$$y_p(t) = t\left[(A_0 t + A_1)\cos(t) + (B_0 t + B_1)\sin(t)\right]$$

You can go on to solve for those coefficients now.

5.2 Method of Variation of Parameters

Start with a normal non-homogeneous second-order linear ODE with variable coefficients:

$$y'' + p(t)y' + q(t)y = g(t) \tag{5.6}$$

We first solve the homogeneous form of the differential equation

$$y_c'' + p(t)y_c' + q(t)y_c = 0 \tag{5.7}$$

to yield the following equations:

$$\begin{aligned} y_c(t) &= c_1 y_1(t) + c_2 y_2(t) \\ y_c'(t) &= c_1 y_1'(t) + c_2 y_2'(t) \end{aligned}$$

Notice here that we get the equation for $y_c'(t)$ by assuming c_1 and c_2 are constants. We will *now* vary the parameters c_1 and c_2 by making them functions of t, hoping that will give us a solution $y(t)$. This gives us the following pair of equations:

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{5.8}$$

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) \tag{5.9}$$

We'll now differentiate equation (5.8) to obtain the following:

$$y'(t) = [u_1'(t)y_1(t) + u_2'(t)y_2(t)] + [u_1(t)y_1'(t) + u_2(t)y_2'(t)]$$

If we look back at equation (5.9), we can see that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \tag{5.10}$$

This relation is called the constraint equation. If we differentiate equation (5.9), we get the following:

$$y''(t) = [u_1(t)y_1''(t) + u_2 y_2''(t)] + [u_1'(t)y_1'(t) + u_2'(t)y_2'(t)] \tag{5.11}$$

Now if we substitute equations (5.8), (5.9), and (5.11) into the original ODE (5.6), we get the following:

$$\underbrace{[y_1'' + p(t)y_1' + q(t)y_1]}_{y_1 \text{ is a solution to (5.7)}} u_1 + \underbrace{[y_2'' + p(t)y_2' + q(t)y_2]}_{y_2 \text{ is a solution to (5.7)}} u_2 + u_1'y_1' + u_2'y_2' = g(t)$$

$$u_1'y_1' + u_2'y_2' = g(t)$$

This gives us the following systems of equations from our previous result and equation (5.10):

$$y_1 u_1' + y_2 u_2' = 0 \tag{5.12}$$

$$y_1' u_1' + y_2' u_2' = g(t) \tag{5.13}$$

This can be written in the following way:

$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

From this we can clearly see that, to solve for u_1' and u_2', the Wronskian must be non-0, as the determinant of the coefficient matrix *is* the Wronskian. To solve for u_1' and u_2', we could just solve the system of equations, or we could use our skills from Linear Algebra and invert the coefficient matrix. This gives us the following:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

This gives us the following equations for $u_1(t)$ and $u_2(t)$:

$$u_1 = -\int \frac{y_2 g(t)}{W(y_1, y_2)} dt$$
$$u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

We can put these back into $y(t)$ now.

$$y(t) = \underbrace{\left[-\int \frac{y_2 g(t)}{W(y_1, y_2)} dt\right] y_1(t) + \left[\int \frac{y_1 g(t)}{W(y_1, y_2)} dt\right] y_2(t)}_{y_p(t)} + \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{y_c(t)}$$

This is the general solution for all non-homogeneous ODEs of variable coefficients when solved with variation of parameters. Remember, when using this result, the ODE you start with *must* be in standard form.

Example

$$y'' - 2y' + y = \frac{e^t}{t}$$

Let's solve for the complementary solution:

$$(r^2 - 2r + 1) = 0$$
$$(r - 1)^2 = 0 \implies r = 1$$
$$y_c(t) = c_1 e^t + c_2 t e^t$$

Let's get the Wronskian:

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t[1 + t] \end{vmatrix} = e^2 t$$

Let's solve for $u_1(t)$ and $u_2(t)$.

$$u_1 = -\int \frac{te^t \cdot e^t}{te^{2t}} dt = -t + c_1'$$
$$u_2 = \int \frac{e^t e^t}{te^{2t}} dt = \ln|t| + c_2'$$

This gives us the following equation for $y(t)$:

$$y(t) = \tilde{c}_1 e^t + \tilde{c}_2 te^t + (-t + c_1')e^t + (\ln|t| + c_2')te^t$$

Reducing this equation, we get our final form:

$$y(t) = c_1 e^t + c_2 te^t + \ln|t| te^t$$

5.3 Mechanical and Electrical Vibrations

Let's consider a mass on a frictionless surface that is attached to a spring and dashpot, and is being acted on by a force. Hooke's Law ($F = -kx$) tells us that the force delivered by the spring is linear in the displacement of the spring from equilibrium[3], and the dashpot is designed to deliver a force proportional to the velocity of the mass. This gives us the following ODE describing the motion of the mass:

$$F = ma = ma_x = \sum F_x$$
$$mu''(t) = \underbrace{-ku(t)}_{\text{Spring force}} - \underbrace{\gamma u'(t)}_{\text{Dashpot drag force}} + \underbrace{F_0 \cos(\omega t)}_{\text{External force}}$$

This yields the general ODE describing the motion of this mass that we will constantly refer back to in this section.

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos(\omega t) \tag{5.14}$$

5.3.1 Undamped, Free Motion

Let's consider the case where $\gamma = 0$ (undamped) and $F_0 = 0$ (free). This gives us the following ODE:

$$r^2 + \frac{k}{m} = 0$$
$$r = \pm\sqrt{\frac{k}{m}} i$$

[3]It actually turns out that this is only valid for small displacements from the spring's equillibrium position. At large displacements, an x^3 term becomes large enough to affect results. We will not worry about these cases in this analysis.

We call $\sqrt{\frac{k}{m}}$ the natural frequency, ω_0. This gives us the following solution for the displacement:

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Let's generate a different way to write this equation. Let's identify the following trigonometric identity:

$$\cos(\omega_0 t - \delta) = \cos(\delta)\cos(\omega_0 t) + \sin(\delta)\sin(\omega_0 t)$$

Let's multiply this by R:

$$R\cos(\omega_0 t - \delta) = \underbrace{R\cos(\delta)}_{A}\cos(\omega_0 t) + \underbrace{R\sin(\delta)}_{B}\sin(\omega_0 t) = u(t)$$

Now the initial conditions are physical quantities: the amplitude and phase shift of the wave. The *period* of oscillation is defined as the time it takes to complete one oscillation. We know that, because $\omega_0 t$ is the argument of the sin and cos functions that define the oscillation, we know that a single period passes when $\omega_0 t = 2\pi$. This gives the period

$$T = \frac{2\pi}{\omega_0}$$

But how do we get R and δ out of initial conditions? Let's differentiate our general solution for $u(t)$:

$$u(t) = R\cos(\omega_0 t - \delta)$$
$$u'(t) = -R\omega_0\sin(\omega_0 t - \delta)$$

Now let's consider some example initial conditions:

Initial Position, No Shove

$$u(0) = u_0; \quad u'(0) = 0$$

Now we can perform the following analysis:

$$u(0) = u_0 = R\cos(-\delta) = R\cos(\delta)$$
$$u'(0) = 0 = -R\omega_0\sin(-\delta) = R\omega_0\sin(\delta)$$

The second equation implies that, if $R \neq 0$, $\delta = 0$. If $R = 0$, we just have a trivial solution. Given $\delta = 0$, we know that $R = u_0$ from the first equation. This gives the following solution to the IVP;

$$u(t) = u_0\cos(\omega_0 t)$$

A Shove

$$u(0) = 0; \quad u'(0) = u'_0$$

Given R is non-0 like in our last example, we can perform the following analysis:

$$u(0) = R\cos(\delta) = 0 \implies \delta = \frac{\pi}{2}$$

$$u'(0) = R\omega_0 \sin(\delta) = R\omega_0 = u'_0 \implies R = \frac{u'_0}{\omega_0}$$

I skipped some of the steps that were more explicit in the last example. This analysis gives us the following equation for position $u(t)$:

$$u(t) = \frac{u'_0}{\omega_0}\cos(\omega_0 t - \frac{\pi}{2})$$

5.3.2 Damped Oscillation, Free Motion

This implies that $\gamma \neq 0$, but $F_0 = 0$. Although it is damped, the motion of the mass is said to be free because there is no external force. This gives the following ODE:

$$mu'' + \gamma u' + ku = 0$$

Solving the characteristic equation, we get the following roots:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

Now we have the same 3 cases for the discriminant that we saw in section 4.4.

Case 1: Real, Distinct Roots We will first consider the case where $\gamma^2 - 4km > 0$. The roots will always be negative. This is because γ is always greater than $\sqrt{\gamma^2 - 4km}$ because γ, k, and m are all strictly positive. We will have the solution to the ODE:

$$u(t) = A\exp\left\{\frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}t\right\} + B\exp\left\{\frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m}t\right\}$$

Because r_1 and r_2 are negative, $u(t)$ will go to 0 as t goes to ∞. This type of motion is called "overdamped motion."

Case 2: Real, Repeated Roots Now, $\gamma^2 = 4km$, giving us the repeated root $r = \frac{-\delta}{2m}$. This gives us the following equation for the motion of the mass:

$$u(t) = A\exp\left\{\frac{-\gamma}{2m}t\right\} + Bt\exp\left\{\frac{-\gamma}{2m}t\right\}$$

This motion is referred to as "critically-damped motion." Also, we know that

$$\gamma = \sqrt{km}$$

in this case.

Case 3: Complex Conjugate Roots In this case, we know that $\gamma^2 - 4km < 0$. This gives us the following equation for the motion of the mass:

$$u(t) = \exp\left\{-\frac{\gamma}{2m}\right\}\left[c_1 \cos\left(\frac{\sqrt{km-\delta^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{km-\delta^2}}{2m}t\right)\right]$$

This motion is called "underdamped motion."

5.3.3 Forced Oscillations

We will be focusing on the case where $\gamma = 0$, $\omega \neq \omega_0$, and $F_0 \neq 0$ in equation (5.14). This gives the following ODE:

$$mu'' + km = F_0 \cos(\omega t)$$

Solving for the complementary solution:

$$\begin{aligned} u_c(t) &= c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \end{aligned}$$

Solving for $u_p(t)$, we get

$$u_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Now let's suppose $u(0) = 0$ and $u'(0) = 0$. This implies that

$$c_1 = \frac{-F_0}{m(\omega^2 - \omega^2)}; \quad c_2 = 0,$$

yielding our solution

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} [\cos(\omega t) - \cos(\omega_0 t)]$$

Now let's consider the trigonometric identity

$$2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

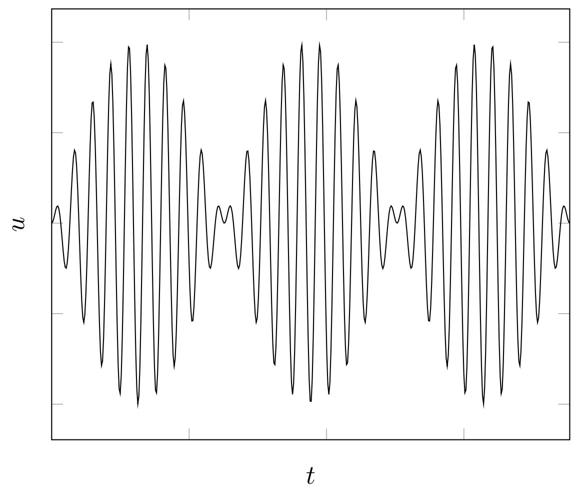
Let $\alpha + \beta = \omega_0 t$ and $\alpha - \beta = \omega t$. This implies

$$\begin{aligned} \alpha &= \frac{t}{2}(\omega_0 + \omega) \\ \beta &= \frac{t}{2}(\omega_0 - \omega) \end{aligned}$$

This means we can rewrite $u(t)$ as

$$u(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left[\frac{(\omega_0 - \omega)}{2}t\right]}_{\text{time-dependent amplitude}} \sin\left[\frac{(\omega_0 + \omega)}{2}t\right]$$

We can think about $\sin\left(\frac{(\omega_0-\omega)}{2}t\right)$ as the time-dependent amplitude because, as ω approaches ω_0, that $\sin\left(\frac{(\omega_0-\omega)}{2}t\right)$ oscillates slowly. In contrast, $\sin\left(\frac{(\omega_0+\omega)}{2}t\right)$ oscillates quickly. When ω approaches ω_0, the graph of $u(t)$ looks similar to the following:



The outer sine wave that seemingly limits the inner sine wave is the slowly oscillating part we have identified as the time-dependent amplitude. This would sound like the same frequency with changing volume. This volume modulation can be used to transmit information, and that is the basis of AM (amplitude modulation) radio. But what happens when ω_0 gets very close to ω? The coefficient in the bottom is a problem. Let's first tackle this with a taylor series. We know that, for small x, $\sin(x) \approx x$. We also know that $\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$. Let's put this into our equation for u:

$$\begin{aligned} u(t) &= \frac{2F_0}{m(\omega_0 - \omega)(\omega_0 + \omega)t} \left(\frac{(\omega_0 - \omega)}{2}t\right) \sin\left[\frac{(\omega_0 + \omega)}{2}t\right] \\ &= \frac{tF_0}{m(\omega_0 + \omega)} \sin\left[\frac{(\omega_0 + \omega)}{2}t\right] \end{aligned}$$

However, to know for sure that this is the behavior, we should solve the ODE again, this time with $\omega_0 = \omega$.

$$\begin{aligned} mu'' + ku &= F_0 \cos(\omega_0 t) \\ u_c(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ u_p(t) &\neq A\cos(\omega_0 t) + B\sin(\omega_0 t) \quad \text{(tricky case!)} \\ u_p(t) &= At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \end{aligned}$$

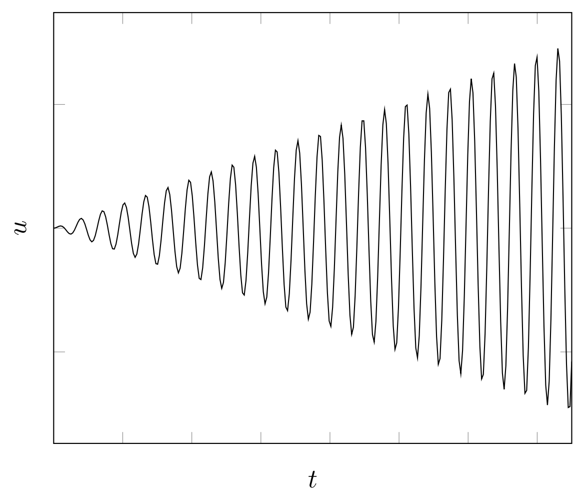
Solving for $u_p(t)$, we get

$$u_p(t) = \frac{F_0 t}{2\omega_0 m} t \sin(\omega_0 t)$$

This gives is our equation describing the motion of the mass:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \underbrace{\frac{F_0}{2\omega_0 m}}_{\text{divergent amplitude}} t \sin(\omega_0 t)$$

A graph of this function will look something similar to the following:



When $\omega = \omega_0$, ω has reached the resonance frequency. Like when you're pushing someone on a swing, when you give an external force with the same frequency as the swing's natural frequency, the amplitude of the swing's motion increases and increases. This should give some intuition as to why the graph of the system when the external force is the same frequency as the system looks the way it does.

5.3.4 Damped, Forced Motion

$$\gamma \neq 0, \quad \text{any } \omega, \quad F_0 \neq 0$$

The damping prevents the solution from diverging. We can't solve for $u(t)$ without knowing more specifics, but no matter what happens, $u_c(t)$ will always go to 0 as t goes to ∞. The particular solution will have the form

$$u_p(t) = R\cos(\omega t - \delta)$$

This doesn't go to 0 as t goes to ∞, so we know that behavior at large t is defined by the following:

$$u(t) \approx R\cos(\omega t - \delta)$$

This is what we call the *time-asymptotic* part of the solution $u(t)$. Some texts call this the "steady-state solution," but we won't call it that because it is not constant. This part of the solution that *does* damp away with time is called the *transient* part of the solution.

6 Higher-Order Linear ODEs

Let's consider the following n^{th}-order ODE as defined by the linear operator L:

$$\begin{aligned} L[y](t) &= y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \ldots + p_{n-1}(t)y^{(1)} + p_n(t)y \\ &= g(t) \end{aligned}$$

The complementary solution will have n fundamental solutions. This means that $y_c(t)$ will have the following form:

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t)$$

Remember how we had, for second-order initial-value problems, we would have initial values for $y(t)$ itself and its first derivative? It's the same for higher-order I-VPs. We will normally have initial values given to the function itself and to all of its derivatives up to the $n-^{\text{st}}$. The initial conditions will have the following form:

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) + \ldots + c_ny_n(t_0) + y_p(t_0) &= y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) + \ldots + c_ny_n'(t_0) + y_p'(t_0) &= y_0' \\ c_1y_1''(t_0) + c_2y_2''(t_0) + \ldots + c_ny_n''(t_0) + y_p''(t_0) &= y_0'' \\ &\vdots \\ c_1y_1^{(n-2)}(t_0) + c_2y_2^{(n-2)}(t_0) + \ldots + c_ny_n^{(n-2)}(t_0) + y_p^{(n-2)}(t_0) &= y_0^{(n-2)} \\ c_1y_1^{(n-1)}(t_0) + c_2y_2^{(n-1)}(t_0) + \ldots + c_ny_n^{(n-1)}(t_0) + y_p^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

This is a linear system of equations, so we can put it in matrix form. I'm going to move the y_p terms over to the right side for simplicity.

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{bmatrix}}_{n \times n \text{ coefficient matrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 - y_p(t_0) \\ y_0' - y_p'(t_0) \\ \vdots \\ y_0^{(n-1)} - y_p^{(n-1)}(t_0) \end{bmatrix}$$

Notice, the coefficient matrix is an $n \times n$ matrix, and it is invertible when its determinant is non-0. The determinant of this matrix is the Wronksian of the fundamental solutions, just as we saw for second-order ODEs. As we say before, the Wronskian must be non-0 if we are to find a solution to this initial-value problem. Let's write our the Wronksian just for clarity:

$$W(y_1, y_2, \dots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Abel's theorem also holds for higher-order ODEs. Given an ODE of the following form:

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \ldots + p_{n-1}(t)y'(t) + p_n(t)y(t) = g(t)$$

The Wronksian is given by the coefficient of the $y^{(n-1)}$ term:

$$W = C \exp\left\{-\int p_1(t)dt\right\}$$

6.1 Homogeneous Linear ODEs of Constant Coefficients

Let's consider the following ODE:

$$L[y](t) = a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = 0 \tag{6.1}$$

We're going to consider, as we have done in the past, solutions in the form of $y = Ce^{rt}$. Remember, we do this because the solutions to these ODEs are translationally-invariant. This means that if you translate the function, the solution will be of the same form. This property can best be shown with the following identity:

$$c_1 e^{r(t+\delta)} = c_1 e^{rt} e^{r\delta} = c_2 e^{rt}$$

The characteristic equation, then, for equation (6.1), is the following:

$$Ce^{rt}\left[c_1 r^n + c_2 r^{n-1} + \ldots + c_n\right] = 0$$

In general, we can't easily solve n^{th}-order polynomials, but in some special situations we can. Let's consider the types of roots we will get when solving these polynomials:

6.1.1 Case 1: Real, Distinct Roots

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

Let's do an example:

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$
$$y(0) = 1; \quad y'(0) = 0; \quad y''(0) = -2; \quad y'''(0) = -1$$

This gives us the following characteristic equation:

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

We know that this factors into the following:

$$(r-1)(r+1)(r-2)(r+3) = 0$$

which gives us the following real, distinct roots:

$$r_1 = 1,\ r_2 = -1,\ r_3 = 2,\ r_4 = -3.$$

This gives us the following solution:

$$y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-3t}$$

Now let's applying the initial conditions. First we need to differentiate $y(t)$ three times.

$$\begin{aligned}
y(t) &= c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-3t} \\
y'(t) &= c_1e^t - c_2e^{-t} + 2c_3e^{2t} - 3c_4e^{-3t} \\
y''(t) &= c_1e^t + c_2e^{-t} + 4c_3e^{2t} + 9c_4e^{-3t} \\
y'''(t) &= c_1e^t - c_2e^{-t} + 8c_3e^{2t} - 27c_4e^{-3t}
\end{aligned}$$

We uses these to generate equations from our initial conditions. I will write this in matrix form to save space.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

We can solve this using methods from Linear Algebra or just standard ways you learned in high school. Solving this system we get the following c values:

$$c_1 = \frac{11}{8},\ c_2 = \frac{5}{12},\ c_3 = -\frac{2}{3},\ c_4 = -\frac{1}{8}$$

and this directly yields our final solution

$$y(t) = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$

So, as you can see, this process is very similar to what we used for second-order homogeneous initial-value problems.

6.1.2 Case 2: Complex Roots

Let's consider the roots

$$r = r_k, r_{k+1} = \lambda \pm \mu i$$

This gives us the associated fundamental solutions:

$$e^{(\lambda+i\mu)t},\ e^{(\lambda-i\mu)t}$$
$$e^{\lambda}\cos(\mu t),\ e^{\lambda}\sin(\mu t)$$

Let's do a simple example:

$$y^{(4)} - y = 0 \tag{6.2}$$

Let's try to perform some analysis on the characteristic equation:

$$\begin{aligned} r^4 - 1 &= 0 \\ r^4 &= 1 \\ r^2 &= \pm 1 \\ r_1 = 1,\ r_2 = -1,\ r_3 &= i,\ r_4 = -i \end{aligned}$$

This gives us the following solution to the ODE:

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3\cos(t) + c_4\sin(t)$$

Now, going back to our original ODE (6.2), let's make a small change. Consider the ODE

$$y^{(4)} + y = 0$$

This gives us the following characteristic equation:

$$r^4 + 1 = 0 \tag{6.3}$$

Doing the same analysis, we will need to figure out what the $\sqrt[4]{-1}$ means. This will require a more in-depth knowledge of complex numbers and their representations.

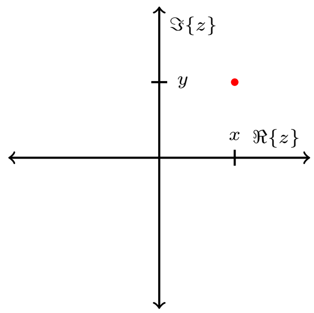
An aside on complex numbers Complex numbers, typically designated the variable name z, can be written in rectangular form in the following way:

$$z = x + iy,\ x, y \in \mathbb{R}.$$

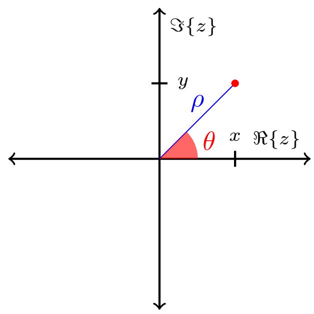
The real part is x, and the imaginary part is y. This is written down mathematically using the following operators:

$$\text{Re}(z) = \Re(z) = x,\ \text{Im}(z) = \Im(z) = y$$

In the complex plane it looks something like this:



We can also see that we can describe this point using a magnitude and an angle. These quantities are typically designated the ρ and θ variable names respectively.



Our good friend Pythagoras has told us that

$$\rho^2 = x^2 + y^2$$
$$\rho = +\sqrt{x^2 + y^2}$$

Keep in mind, because ρ is a radial quantity, it is always positive. If you negated ρ, that new complex number could be represented just be changing the angle by adding π. We also know that, by definition of tangent, that

$$\tan(\theta) = \frac{y}{x}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

And we know from the definition of sin and cos that

$$x = \rho\cos(\theta)$$
$$y = \rho\sin(\theta)$$

This means that we can rewrite z the following way:

$$\begin{aligned} z &= x + iy \\ &= \rho\cos(\theta) + i\rho\sin(\theta) \\ &= \rho[\cos(\theta) + i\sin(\theta)] \end{aligned}$$

Recall Euler's formula. Using it, we can reduce z into the following form:

$$z = \rho e^{i\theta}$$

This is the form we wanted: z parameterized by a magnitude ρ and an angle θ. This form is referred to as the *polar* form or representation of complex numbers. Keep in mind that θ can have any integer multiple of 2π added to it, and the resulting complex number will not be changed. There is a better way to write ρ. We can write the complex conjugate of z as $\bar{z}$.

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \end{aligned}$$

The $|z|^2$, the square of the magnitude of z, is defined as

$$\begin{aligned} |z|^2 &= z\bar{z} \\ &= (x + iy)(x - iy) \\ &= x^2 + ixy - ixy + y^2 \\ &= x^2 + y^2 \\ |z| &= \sqrt{x^2 + y^2} \\ &= \rho \end{aligned}$$

This means that we can rewrite z as:

$$z = |z|e^{i\theta}$$

Also, notice that the complex conjugate of z negates the angle in this form:

$$z = |z|e^{-i\theta}$$

End of aside

Keeping in mind the polar form of complex numbers, let's try to evaluate $\sqrt[4]{-1}$.

Roots of (Negative) Unity First we need to be able to represent -1 as a complex number. Using the definitions of θ and ρ, we can get their values:

$$\begin{aligned} \theta &= \arctan\left(\frac{y}{x}\right) = \arctan(0) = 0 \\ \rho &= \pm\sqrt{x^2 + y^2} = \pm\sqrt{1} = \pm 1 \end{aligned}$$

Here we know that ρ must be positive, $\rho = 1$ with $\theta = 0$ gives us 1 not -1. We get -1 when $\rho = -1$, but ρ cannot be negative. We can make safely force ρ to be positive by adding π to the angle θ. Adding π to our angle (0) and forcing ρ to be positive, we get

$$-1 = e^{i\pi}$$

We also know that we can add any integer multiple of 2π to our angle theta, and we get the same number.

$$-1 = e^{i(\pi + 2\pi m)},\ \forall m \in \mathbb{Z}$$

Now we can try to find the fourth roots of -1. We can do this using our complex representation of -1:

$$\sqrt[4]{-1} = e^{i(\pi + 2\pi m)^{\frac{1}{4}}} = e^{i(\frac{\pi}{4} + \frac{\pi}{2} m)}$$

Let's put in some values for m:

$m = 0$: $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m = 1$: $e^{i\frac{\pi}{4} + \frac{\pi}{2}} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m = 2$: $e^{i\frac{\pi}{4} + \pi} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$m = 3$: $e^{i\frac{\pi}{4} + \frac{3\pi}{2}} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$m = 4$: $e^{i\frac{\pi}{4} + 2\pi} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$
We see that this is root is already accounted for with $m = 0$, and we can easily see that because we have just added 2π the angle associated with $m = 0$.

It can be seen that there are always n n^{th} roots of any given number. In our example, because we are taking the fourth root of -1, there are going to be four distinct roots. Taking the roots of the characteristic equation from (6.3) to be equal to these distinct roots of -1 we get the following:

$$r_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$r_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$r_3 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$
$$r_4 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

This gives us the following solution to the ODE

$$y(t) = e^{\frac{t}{\sqrt{2}}}\left[c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right)\right] + e^{\frac{-t}{\sqrt{2}}}\left[c_3 \cos\left(\frac{t}{\sqrt{2}}\right) + c_4 \sin\left(\frac{t}{\sqrt{2}}\right)\right]$$

The characteristic equation of an ODE with real coefficients will only ever have real or complex conjugate roots. There will never be an unpaired complex root.

Now let's consider the following ODE:

$$y^{(4)} + 16y = 0$$
$$r^4 + 16 = 0 \implies r = (-16)^{\frac{1}{4}} = \left|\sqrt[4]{16}\right| \sqrt[4]{-1} = 2\sqrt[4]{-1}$$

We just found the fourth roots of -1, so we can just multiply all those roots by 2. This gives us the following ODE:

$$y(t) = e^{t\sqrt{2}}\left[c_1 \cos\left(t\sqrt{2}\right) + c_2 \sin\left(t\sqrt{2}\right)\right] + e^{-t\sqrt{2}}\left[c_3 \cos\left(t\sqrt{2}\right) + c_4 \sin\left(t\sqrt{2}\right)\right]$$

6.1.3 Case 3: Repeated Roots

If the k^{th} root r_k^* appears s times and is real

$$y_s^k(t) = e^{r_k^* t},\ te^{r_k^* t},\ t^2 e^{r_k^* t}$$

If the roots are complex conjugate pairs $\lambda + i\mu$ and $\lambda - i\mu$ each appear s times, then the fundamental solutions will have the form:

$$y_s^k(t) = e^{\lambda t}\sin(\mu t), e^{\lambda t}\cos(\mu t);\ te^{\lambda t}\cos(\mu t), te^{\lambda t}\sin(\mu t);$$
$$t^2 e^{\lambda t}\cos(\mu t), t^2 e^{\lambda t}\sin(\mu t), \ldots, t^{s-1}e^{\lambda t}\cos(\mu t), t^{s-1}e^{\lambda t}\sin(\mu t)$$

Here the k represents the index of the root, not an exponent. Let's do an example problem:

$$y''' - 3y'' + 3y' - y = 0$$
$$r^3 - 3r^2 + 3r - 1 = 0$$
$$(r-1)^3 = 0 \implies r_1 = r_2 = r_3 = 1$$

This gives the following solution

$$y(t) = c_1 e^t + c_2 te^t + c_2 t^2 e^t$$

6.2 Non-Homogeneous Higher-Order ODEs

6.2.1 Method of Undetermined Coefficients

$$L[y](t) = a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_{n-1}y'(t) + a_n y(t) = g(t)$$

The method of Undetermined Coefficients with higher-order ODEs is completely analogous to what we did with second-order ODEs. Let's do an example:

$$y''' - 6y'' + 11y' - 6y = 2te^{-t}$$
$$r^3 - 6r^2 + 11r - 6 = 0$$

Now let's seriously factor this cubic polynomial. The -6 on the far right, because it is a product of the roots, tells us that it is likely that $\pm1, \pm2, \pm3$, and ±6 are likely roots. Putting +1 in, we see that it does solve the equation. Now we must divide out 1. We will do this by way of synthetic division:

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array} \implies (r-1)(r^2-5r+6)=0$$

Synthetic division works like this: put all the coefficients on the top (careful not to leave out any zeros for missing terms). Put the root on the left (in this case 1), and then bring down the first coefficient (also 1). Now multiply the root by the number on the bottom, yielding the next number to put below the -6: 1 ($1 \times 1 = 1$). Then add those two numbers ($-6+1=-5$). Now repeat what you have just done. Multiply the new number (-5) by the root (1) to produce our next number (-5). Then we add 11 to this, yielding 6. We keep going, and importantly we end with 0 at the end. If we didn't end with a 0, then 1 isn't be a root. After that, we can just read off the result. If you need more practice with synthetic division, look online.

Going back to our example, we can easily see that

$$(r-1)(r^2-5r+6) = (r-1)(r-2)(r-3) \implies r_1 = 1,\ r_2 = 2,\ r_3 = 3$$
$$y_c(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$$

We also see that we can choose the particular solution to be

$$y_p(t) = [A_0t + A_1]\,e^{-t}$$

with no overlap with the complementary solution. Differentiating this and putting back into the ODE, we get the following equation

$$-24A_0te^{-t} + (26A_0 - 24A_1)e^{-t} = 2te^{-t}$$

Solving for $A_0 = -\frac{1}{12}$ and $A_1 = -\frac{13}{144}$, we get the following particular solution:

$$y_p(t) = -\frac{1}{12}te^{-t} - \frac{13}{144}e^{-t}$$

More "Tricky Cases" We have the same sort of tricky cases we had for second-order ODES, just instead of having just A1, A2, B1, and B2, we have A1 through AN and B1 through BN. We will again use $g_n(t)$ to represent each term of $g(t)$. Let's start with the A cases:

A1. The y term is gone: *one* zero root, so we multiply what we would otherwise use by t.

A2. The y' and y terms are gone: *two* zero roots, so we multiply what we would other use by t^2.

A3. Follow the pattern: we have three zero roots, multiply what we would otherwise use by t^3.

AN. We have n zero roots, and we multiply what we would otherwise use by t^n.

As we already know from our previous discussion on tricky cases in the context of second-order ODEs, the A cases are just special cases of the B cases.

B1. If $g_n(t)$ or its derivatives are proportional to a fundamental solution resulting from a root that is *not* repeated, we multiply what we would otherwise use by t.

B2. If $g_n(t)$ or its derivatives are proportional to a fundamental solution resulting from a root that appears exactly *twice*, we multiply what we would otherwise use by t^2.

B3. If $g_n(t)$ or its derivatives are proportional to a fundamental solution resulting from a root that appears exactly *three* times, we multiply what we would otherwise use by t^3.

BN. If $g_n(t)$ or its derivatives are proportional to a fundamental solution resulting from a root that appears exactly n times, we multiple what we would otherwise use by t^n.

This long, verbose explanation for what we do in each case can be simplified as we did before for second-order ODEs. What we really have to worry about is just multiplying our $g_n(t)$ terms and their derivatives by t enough times such that we don't have any overlap with the fundamental solutions. Let's do an example:

$$\begin{aligned} y''' - 4y'' - 4y' &= t + 3\cos(t) + e^{2t} \\ r(r^2 - 4r + 4) &= 0 \\ r(r-2)^2 &= 0 \implies r_1 = 0,\ r_2 = 2,\ r_3 = 2 \\ y_c(t) &= c_1 + c_2 e^{2t} + c_3 t e^{2t} \end{aligned}$$

Let's look at each individual term for $g(t)$:

t: Differentiating this gives us t and 1. We can see that 1 appears as a fundamental solution (c_1 in our expression for $y_c(t)$). This means we must multiply what we would otherwise use by t.

$$t\left[A_0 t + A_1\right]$$

$\cos(t)$: Differentiating this gives us $\cos(t)$ and $\sin(t)$. We can see that neither of those terms overlap with the fundamental solution, so we don't multiply anything extra.

$$\left[B_0\cos(t) + B_1\sin(t)\right]$$

e^{2t}: Differentiating this just gives us e^{2t}. We can see that this appears in the fundamental solution, but even after we multiply it by t, we still haven't resolved the conflict. So we multiply it by t again, finally getting a term that isn't already covered by the fundamental solutions:

$$t^2\left[C_0 e^{2t}\right]$$

Putting all of these terms together, we get a form for $y_p(t)$:

$$y_p(t) = t\left[A_0 t + A_1\right] + \left[B_0\cos(t) + B_1\sin(t)\right] + t^2\left[C_0 e^{2t}\right]$$

6.2.2 Method of Variation of Parameters

The method of variation of parameters works for any linear non-homogeneous ODE with variable coefficients.

$$L[y](t) = y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \ldots + p_{n-1}(t)y'(t) + p_n(t)y(t) = g(t) \qquad (6.4)$$

We will have n fundamental solutions, giving us the following form for the complementary solution:

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \ldots c_ny_n(t)$$

The basic idea is the same as the one we had for second-order ODEs.

1. Solve $L[y](t) = 0$ giving n fundamental solutions, checking the Wronksian.
2. Write the derivatives of $y_c(t)$ up to the $(n-1)^{\text{st}}$ derivative. I will rewrite c_n as c_m and y_n as y_m because the coefficients and the functions we use don't depend on how much we've differentiated $y(t)$.

$$\begin{aligned}
y_c(t) &= c_1y_1(t) + c_2y_2(t) + \ldots + c_my_m(t)\\
y_c'(t) &= c_1y_1'(t) + c_2y_2'(t) + \ldots + c_my_m'(t)\\
y_c''(t) &= c_1y_1''(t) + c_2y_2''(t) + \ldots + c_my_m''(t)\\
\vdots &= \quad \vdots \quad + \quad \vdots\\
y_c^{(n-1)}(t) &= c_1y_1^{(n-1)}(t) + c_2y_2^{(n-1)}(t) + \ldots + c_my_m^{(n-1)}(t)
\end{aligned}$$

3. Now we vary our parameters. $c_i \to u_i(t), i \in \{1, \ldots m\}$.

$$\begin{aligned}
y_c(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) + \ldots + u_m(t)y_m(t)\\
y_c'(t) &= u_1(t)y_1'(t) + u_2(t)y_2'(t) + \ldots + u_m(t)y_m'(t)\\
y_c''(t) &= u_1(t)y_1''(t) + u_2(t)y_2''(t) + \ldots + u_m(t)y_m''(t)\\
\vdots &= \quad \vdots \quad + \quad \vdots\\
y_c^{(n-1)}(t) &= u_1(t)y_1^{(n-1)}(t) + u_2(t)y_2^{(n-1)}(t) + \ldots + u_m(t)y_m^{(n-1)}(t)
\end{aligned}$$

4. Differentiate each of the equation to get an expression for the one below it (excluding the expression for $y_c^{(n-1)}$) to generate our contraint equations.

$$\begin{aligned}
\frac{d}{dt}y_c(t) &= \left[u_1(t)y_1'(t) + \cdots + u_m(t)y_m'(t)\right] + \left[u_1'(t)y_1(t) + \cdots + u_m'(t)y_m(t)\right]\\
&= y_c'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + \ldots + u_m(t)y_m'(t)\\
&\implies \underbrace{\left[u_1'(t)y_1(t) + \cdots + u_m'(t)y_m(t)\right] = 0}_{\text{first constraint equation}}
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt} y_c'(t) &= [u_1(t)y_1''(t) + \cdots + u_m(t)y_m''(t)] + [u_1'(t)y_1'(t) + \cdots + u_m'(t)y_m'(t)] \\
&= y_c''(t) = u_1(t)y_1''(t) + u_2(t)y_2''(t) + \ldots + u_m(t)y_m''(t) \\
&\implies \underbrace{[u_1'(t)y_1'(t) + \cdots + u_m'(t)y_m'(t)] = 0}_{\text{second constraint equation}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt} y_c^{(n-2)}(t) &= \left[u_1(t)y_1^{(n-1)}(t) + \cdots + u_m(t)y_m^{(n-1)}(t)\right] \\
&\quad + \left[u_1'(t)y_1^{(n-2)}(t) + \cdots + u_m'(t)y_m^{(n-2)}(t)\right] \\
&= y_c^{(n-1)} = u_1(t)y_1^{(n-1)}(t) + u_2(t)y_2^{(n-1)}(t) + \ldots + u_m(t)y_m^{(n-1)}(t) \\
&\implies \underbrace{\left[u_1'(t)y_1^{(n-2)}(t) + \cdots + u_m'(t)y_m^{(n-2)}(t)\right] = 0}_{(n-2)^{\text{nd}} \text{ constraint equation}}
\end{aligned}
$$

This gives us $n-1$ constraint equations, but we have to solve for n coefficients: $u_1(t)$ through $u_m(t)$. So where are we going to get the final equation?

We are solving a non-homogeneous ODE, and none of our constrant equations have anything to do with $g(t)$. We must include $g(t)$ in our final equation. We can do this by saying our solution satisfies the original ODE. We are going to substitute our derivates for $y_c(t)$ into the original ODE, and include $y_c^{(n)}$ by differeniating our expression for $y_c^{(n-1)}$ when we initially varied our parameters.

$$
\begin{aligned}
y_c^{(n)}(t) &= \frac{d}{dt} y^{(n-1)}(t) = \frac{d}{dt}\left[u_1(t)y_1^{(n-1)}(t) + \ldots + u_m(t)y_m^{(n-1)}(t)\right] \\
&= \left[u_1(t)y_1^{(n)}(t) + \ldots + u_m(t)y_m^{(n)}(t)\right] \\
&\quad + \left[u_1'(t)y_1^{(n-1)}(t) + \ldots + u_m'(t)y_m^{(n-1)}(t)\right]
\end{aligned}
$$

We put all of our equations for y_c and its derivatives back into our original ODE.

$$
y_c^{(n)} + p_1(t)y_c^{(n-1)} + \ldots + p_{n-1}(t)y_c' + p_n(t)y = g(t)
$$

Doing this analysis, we end up with our final equation:

$$
y_1^{(n-1)}u_1'(t) + \ldots + y_m^{(n-1)}(t)u_m'(t) = g(t)
$$

Now we can put all these equations into a matrix:

$$
\begin{bmatrix}
y_1(t) & y_2(t) & \cdots & y_m(t) \\
y_1'(t) & y_2'(t) & \cdots & y_m'(t) \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_m^{(n-2)}(t) \\
y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_m^{(n-1)}(t)
\end{bmatrix}
\begin{bmatrix}
u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{m-1}'(t) \\ u_m'(t)
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ g(t)
\end{bmatrix}
$$

We can use Cramer's Rule from Linear Algebra to solve for each $u'_m(t)$:

$$u'_m(t) = \frac{\widetilde{W}_m(t)}{W\left(y_1, y_2, \ldots, y_n\right)}, \;\; m = 1, 2, 3, \ldots, n$$

where $\widetilde{W}_m(t)$ is defined as the determinant of the matrix where the m^{th} column is replaced with the solution vector. For example: $\widetilde{W}_1(t)$ would be the following:

$$\widetilde{W}_1(t) = \begin{vmatrix} 0 & y_2(t) & \cdots & y_m(t) \\ 0 & y'_2(t) & \cdots & y'_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_2^{(n-2)}(t) & \cdots & y_m^{(n-2)}(t) \\ g(t) & y_2^{(n-1)}(t) & \cdots & y_m^{(n-1)}(t) \end{vmatrix}$$

Normally we take out the $g(t)$ from the determinant and just place it in the numerator like this:

$$u'_m(t) = \frac{g(t)W_m(t)}{W\left(y_1, y_2, \ldots, y_n\right)}, \;\; m = 1, 2, 3, \ldots, n$$

Here $W_m(t)$ is defined as before, but we have taken out $g(t)$ from the vector that was place in the Wronksian. For example: $W_1(t)$ would be the following:

$$W_1(t) = \begin{vmatrix} 0 & y_2(t) & \cdots & y_m(t) \\ 0 & y'_2(t) & \cdots & y'_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_2^{(n-2)}(t) & \cdots & y_m^{(n-2)}(t) \\ 1 & y_2^{(n-1)}(t) & \cdots & y_m^{(n-1)}(t) \end{vmatrix}$$

This gives the following for $u_m(t)$:

$$u_m(t) = \int \frac{g(t)W_m(t)}{W\left(y_1, y_2, \ldots, y_n\right)} dt + c$$

The constant of integration represents the complementary solution $y_m(t)$. This leaves the rest as the complementary solution.

$$y(t) = \underbrace{\sum_{m=1}^{n} c_m y_m(t)}_{y_c(t)} + \underbrace{\sum_{m=1}^{n} \left[y_m(t) \int \frac{g(t)W_m(t)}{W\left(y_1, y_2, \ldots, y_n\right)} dt \right]}_{y_p(t)}$$

Example

$$\begin{aligned} y''' + y' &= \sec(t) \\ r(r^2 + 1) = 0 &\implies r = 0, \pm i \end{aligned}$$

This gives the following fundamental solutions:

$$y_1(t) = 1, \quad y_2(t) = \cos(t), \quad y_3(t) = \sin(t)$$

Now we must solve for the Wronksian to start finding our particular solution.

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \cdot \left[\sin^2(t) + \cos^2(t)\right] = 1$$

Now we must solve for each W_1, W_2, and W_3:

$$\begin{aligned} W_1 &= \begin{vmatrix} 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 1 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \cdot \left[\cos^2(t) + \sin^2(t)\right] = 1 \\ W_2 &= \begin{vmatrix} 1 & 0 & \sin(t) \\ 0 & 0 & \cos(t) \\ 0 & 1 & -\sin(t) \end{vmatrix} = -\cos(t) \\ W_3 &= \begin{vmatrix} 1 & \cos(t) & 0 \\ 0 & -\sin(t) & 0 \\ 0 & -\cos(t) & 1 \end{vmatrix} = -\sin(t) \end{aligned}$$

Now we can solve for each u_1, u_2, and u_3:

$$\begin{aligned} u_1(t) &= \int \frac{W_1 g(t)}{W(y_1, y_2, y_3)} = \int \sec(t) = \ln|\tan(t) + \sec(t)| + c_1 \\ u_2(t) &= \int \frac{W_2 g(t)}{W(y_1, y_2, y_3)} = \int -1 = -t + c_2 \\ u_3(t) &= \int \frac{W_3 g(t)}{W(y_1, y_2, y_3)} = \int -\tan(t) = -\ln|\cos(t)| + c_3 \end{aligned}$$

This gives us the following solution to the ODE:

$$\begin{aligned} y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) + u_3 y_3(t) \\ &= 1 \cdot \big[\ln|\tan(t) + \sec(t)| + c_1\big] + \cos(t)\big[-t + c_2\big] + \sin(t)\big[-\ln|\cos(t)| + c_3\big] \\ &= \underbrace{c_1 + c_2\cos(t) + c_3\sin(t)}_{y_c(t)} + \underbrace{\ln|\tan(t) + \sec(t)| - t\cos(t) - \ln|\cos(t)|\sin(t)}_{y_p(t)} \end{aligned}$$

7 Laplace Transforms

What if we could solve initial-value problems without actually solving differential equations? Such a process exists. We can perform a transformation of our equation, solve what we need to, and then transform back. One type of transformation is the

Laplace transform, a type of integral transformation. Integral transformations are defined in the following way:

$$T\{f(t)\} = F(s) = \int_{\alpha}^{\beta} \underbrace{k(s,t)}_{\text{kernel}} f(t)dt$$

The Laplace transform has bounds $\alpha = 0$ and $\beta = \infty$ with the kernel e^{-st}. The Laplace transform is denoted using a script L: $\mathscr{L}$. It is therefore defined the following way:

$$\mathscr{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)dt \tag{7.1}$$

Here we are transformation the function $f(t)$ from the time domain into the 'Laplace' or 's' domain. We will get into solving ODEs with this transformation a little later; first we need to get a solid grasp of how to work with this.

7.1 Performing the Laplace Transform

7.1.1 Transformable Functions

In equation (7.1), we can see that, if we want the integrand to be integrable, we need it to converge. The domain of the Laplace transform $f(t)$ can be piecewise continuous for $0 \leq t < A$ where A is positive and finite. That means we can only have finite discontinuities, and the function cannot diverge to infinity inside that interval. Beyond $f(A)$, the function $f(t)$ may diverge as $t \to \infty$, but it must do no faster than e^{at}, where $a > s$. Most functions diverge slower than the "big winner" e^{at}. Polynomials and logarithms both diverge slower than e^{at}, but singularities like those that occur in functions such as $(t-1)^{-2}$ and $\tan(t)$ both diverge faster than e^{at} and, therefore, such functions cannot be transformed. It might be best to go through some examples to best grasp this concept:

$$\begin{aligned}
f(t) &= 2e^{100t} &&\longrightarrow && F(s) \text{ exists for } s > 100\\
f(t) &= 2e^{100t^2} &&\longrightarrow && F(s) \text{ does not exist}\\
f(t) &= t^2 + t &&\longrightarrow && F(s) \text{ exists for } s > 0\\
f(t) &= (t-8)^{-1}e^{t} &&\longrightarrow && F(s) \text{ does not exist}\\
f(t) &= (t+8)^{-2} &&\longrightarrow && F(s) \text{ exists for } s > 0\\
f(t) &= (t+8)^{-2}e^{2t} &&\longrightarrow && F(s) \text{ exists for } s > 2\\
f(t) &= 10^{100}e^{te^{74}} &&\longrightarrow && F(s) \text{ exists for } s > e^{74}\\
f(t) &= 2^8\cos(90t)e^{at} &&\longrightarrow && F(s) \text{ exists for } s > a\\
f(t) &= 2^8\tan(90t)e^{at} &&\longrightarrow && F(s) \text{ does not exist}
\end{aligned}$$

Make sure each of these examples make sense.

7.1.2 Improper Integrals

We can see that the Laplace transform requires evaluating an integral with an upper limit of ∞. Integrals with either one or both limits being either positive or negative infinity are said to be "improper." We evaluate them like this

$$\int_{\alpha}^{\infty} f(t)dt = \lim_{R\to\infty} \int_{\alpha}^{R} f(t)dt$$

7.1.3 Examples

Let's try to actually evaluate some Laplace transforms. First let's try $f(t) = 1$, one of the easiest.

$$\mathscr{L}\{1\} = \int_0^{\infty} e^{-st}dt = \lim_{R\to\infty} \int_0^R e^{-st}dt$$

Here we can see that s must be positive for this integral to converge.

$$\begin{aligned}
\mathscr{L}\{1\} &= \lim_{R\to\infty} -\frac{1}{s}e^{-st}\Big|_{t=0}^{R} \\
&= \lim_{R\to\infty} -\frac{1}{s}\left[e^{-sR} - e^0\right] \\
&= -\frac{1}{s}\left[-e^0\right] \\
&= \frac{1}{s}; \quad s > 0
\end{aligned}$$

Now let's try something a little more complicated: $f(t) = t^2$.

$$\mathscr{L}\left\{t^2\right\} = \int_0^{\infty} e^{-st}t^2dt = \lim_{R\to\infty} \int_0^R e^{-st}t^2dt$$

Immediately we see that $s > 0$ for this integral to converge. Here we make use of tabular integration by parts. We make a table of three things: a sign, the derivative column, and the integration column. We are going to put the polynomial in the derivative column because it will eventually go away, and then we throw the exponential in the integration column.

$+$	t^2	e^{-st}
$-$	$2t$	$-\frac{1}{s}e^{-st}$
$+$	2	$\frac{1}{s^2}e^{-st}$
$-$	0	$-\frac{1}{s^3}e^{-st}$

Now we read going across and then down, evaluating the function at our original integral bounds. This yields the following result from the integration:

$$\begin{aligned}\mathscr{L}\left\{t^2\right\} &= \lim_{R\to\infty}\left[-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right]\Bigg|_{t=0}^{R} \\ &= \frac{2}{s^3}; \quad s > 0\end{aligned}$$

In general, the Laplace transform of a polynomial t^n is the following:

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}; \quad s > 0$$

Notice that when $n = 1$, the input function degenerates to 1, and the result matches what we previously calculated.

Let's do one more important example: $f(t) = e^{at}$.

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} &= \int_0^\infty e^{at}e^{-st}dt \\ &= \int_0^\infty e^{(a-s)t}dt \\ &= \frac{1}{a-s}\int_0^\infty e^{(a-s)t}dt \\ &= \frac{1}{s-a}; \quad s > a \qquad (7.2)\end{aligned}$$

This will come in handy later.

7.1.4 Linearity of the Laplace Transform

All integral transformations are linear transformations. Because the Laplace transform is an integral transformation, it must also be linear. That means it satisfies the following identity where c_1 and c_2 are constants and f_1 and f_2 are functions:

$$\mathscr{L}\left\{c_1 f(t) + c_2 f(t)\right\} = c_1\mathscr{L}\left\{f_1(t)\right\} + c_2\mathscr{L}\left\{f_2(t)\right\}$$

With this information, let's try to evaluate the Laplace transform of a three-term polynomial:

$$\begin{aligned}\mathscr{L}\left\{t^5 + 3t^4 + 4\right\} &= \mathscr{L}\left\{t^5\right\} + 3\mathscr{L}\left\{t^4\right\} + 4\mathscr{L}\left\{1\right\} \\ &= \frac{5!}{s^6} + 3\frac{4!}{s^5} + \frac{4}{s} \\ &= \frac{120}{s^6} + \frac{72}{s^5} + \frac{4}{s}\end{aligned}$$

The linearity of the Laplace transform also means that 0 will always map to 0 under the transform.

$$\mathscr{L}\left\{0\right\} = 0$$

7.1.5 Trigonometric Laplace Transforms

It is really important for us to be able to get the Laplace transform of $\sin(at)$ and $\cos(bt)$, because we know it is possible to get them. Let's start with $\sin(at)$:

$$\mathscr{L}\left\{\sin(at)\right\} = F(s) = \int_0^\infty e^{-st}\sin(at)dt$$

Again we see that $s > 0$ for the integral to converge. To evaluate this integral, we are going to use the explicit form of integration by parts. For each step, we integrate the exponential and differentiate the trigonometric function.

$$\begin{aligned}
&= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{\infty} + \int_0^\infty \frac{1}{s}e^{-st}a\cos(at)dt \\
&= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{\infty} + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt \\
&= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{\infty} + \frac{a}{s}\left[-\frac{1}{s}e^{-st}\cos(at)\Big|_{t=0}^{\infty} - \int_0^\infty \frac{1}{s}e^{-st}a\sin(at)dt\right] \\
&= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{\infty} + \frac{a}{s}\left[-\frac{1}{s}e^{-st}\cos(at)\Big|_{t=0}^{\infty} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right]
\end{aligned}$$

Let's evaluate everything that can be evaluated right now.

$$= \frac{a}{s^2} - \frac{a^2}{s^2}\underbrace{\int_0^\infty e^{-st}\sin(at)dt}_{F(s)}$$

When the original integral shows up in the result, the IBP integration is often said to be 'cyclic.' Let's continue with this 'cyclic' integration by parts.

$$\begin{aligned}
&= \frac{a}{s^2}\left[1 - aF(s)\right] = F(s) \\
&\implies a - a^2F(s) = s^2F(s) \\
&\implies a = F(s)\left[s^2 + a^2\right] \\
&\implies F(s) = \mathscr{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2} \qquad (7.3)
\end{aligned}$$

The same process can be done for $\cos(at)$, yielding the following result:

$$\mathscr{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2}$$

Now let's talk about the hyperbolic sin and cos functions: sinh and cosh. Recall how the normal sin and cos functions can be defined in terms of complex exponentials:

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

Their hyperbolic counterparts are defined very similarly, just with the is removed:

$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$
$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

We will use these definitions to find their Laplace transforms. Let's start with $\sinh(at)$.

$$\begin{aligned}
\mathscr{L}\left\{\sinh(at)\right\} &= \frac{1}{2}\left[\mathscr{L}\left\{e^{at}\right\} - \mathscr{L}\left\{e^{-at}\right\}\right] \\
&= \frac{1}{2}\left[\int_0^\infty e^{-st}e^{at}dt - \int_0^\infty e^{-at}e^{-st}dt\right] \\
&= \frac{1}{2}\left[\int_0^\infty e^{(-s+a)t}dt - \int_0^\infty e^{(-s-a)t}dt\right]
\end{aligned}$$

To get both of these integrals to converge, we can see that we need the condition $s > |a|$.

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{-s+a}e^{(-s+a)t}\Big|_{t=0}^{\infty} - \frac{1}{-s-a}e^{(-s-a)t}\Big|_{t=0}^{\infty}\right] \\
&= \frac{1}{2}\left[-\frac{1}{-s+a} + \frac{1}{-s-a}\right] \\
&= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \\
&= \frac{1}{2}\left[\frac{s+a}{s^2-a^2} - \frac{s-a}{s^2-a^2}\right] \\
&= \frac{a}{s^2-a^2}; \quad s > |a|
\end{aligned}$$

Notice that this is very similar to our result for $\mathscr{L}\left\{\sin(at)\right\}$ (7.3), just plus sign in that result is flipped for sinh and the constraint on s is different. Similarly, the result for $\mathscr{L}\left\{\cosh(at)\right\}$ is the following:

$$\mathscr{L}\left\{\cosh(at)\right\} = \frac{s}{s^2-a^2}; \quad s > |a|$$

7.2 Solving Initial-Value Problems

We wouldn't be studying Laplace transforms if we couldn't use it to solve differential equations. We already know that $\mathscr{L}\left\{f(t)\right\} = F(s)$ by definition. What about $\mathscr{L}\left\{f'(t)\right\}$?

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st}f'(t)dt$$

We can solve this using integration by parts.

$$= e^{-st}y(t)\Big|_{t=0}^{\infty} - (-s)\underbrace{\int_0^{\infty} e^{-st}y(t)dt}_{F(s)}$$
$$= -y(0) + sF(s)$$

Alright that's helpful. What about $f''(t)$?

$$\begin{aligned}
\mathscr{L}\{f''(t)\} &= \int_0^{\infty} e^{-st}f''(t)dt \\
&= e^{-st}y'(t)\Big|_{t=0}^{\infty} + s\underbrace{\int_0^{\infty} e^{-st}y'(t)dt}_{\mathscr{L}}\{f'(t)\} \\
&= -y'(0) + s\left[-y(0) + sF(s)\right] \\
&= s^2F(s) - sy(0) - y'(0)
\end{aligned}$$

This is enough to solve second order ODEs. The general form for $\mathscr{L}\left\{f^{(n)}(t)\right\}$ is the following:

$$\begin{aligned}
\mathscr{L}\left\{f^{(n)}(t)\right\} &= s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0) \\
&= s^nF(s) - \sum_{i=0}^{n-1} s^{(n-1)-i}f^{(i)}(0)
\end{aligned}$$

We can use this result to solve ODEs of any order. Let's do an example.

$$y'' - y' - 2y = 0; \qquad y(0) = 1, \quad y'(0) = 0$$

Let's transform both sides.

$$\begin{aligned}
\mathscr{L}\{y''\} - \mathscr{L}\{y'\} - 2\mathscr{L}\{y\} = \mathscr{L}\{0\} &= 0 \\
s^2F(s) - sy(0) - y'(0) - [sF(s) - y(0)] - 2F(s) &= 0 \\
s^2F(s) - s - sF(s) + 1 - 2F(s) &= 0 \\
F(s)\left[s^2 - s - 2\right] &= s - 1 \\
F(s) &= \frac{s-1}{s^2 - s - 2} \\
F(s) &= \frac{s-1}{(s-2)(s+1)}
\end{aligned}$$

Now we need to change $F(s)$ back to $f(t)$. We do this by using the inverse Laplace transform. The way we will invert Laplace transforms in these notes is just by recalling the results of the Laplace transforms we have already calculated. The important Laplace transform for this problem is the following:

$$\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a} \implies \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{7.4}$$

Here we cannot continue without employing partial fractions. Partial fractions will come up frequently when solving ODEs with the Laplace transform.

$$\begin{aligned}
\frac{s-1}{(s-2)(s+1)} &= \frac{A}{s-2} + \frac{B}{s+1} \\
s-1 &= A(s+1) + B(s-2) \\
s-1 &= s(A+B) + (A-2B) \\
&\implies A+B = 1 \text{ and } A-2B = -1 \\
&\implies A = \frac{1}{3}, \quad B = \frac{2}{3} \\
&\implies \frac{s-1}{(s-2)(s+1)} = \frac{1}{3(s-2)} + \frac{2}{3(s+1)}
\end{aligned}$$

Finally let's invert $F(s)$.

$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}\left\{\frac{1}{3} \times \frac{1}{s-2} + \frac{2}{3} \times \frac{1}{s+1}\right\} \\
&= \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}
\end{aligned}$$

Recall equation (7.4).

$$f(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Notice how there are no constants of integration. This is because the Laplace transform we did already made use of the initial values, and we wouldn't have been able to do our analysis without them.

7.3 Solving IVPs Containing Piecewise Functions

To solve IVPs with piecewise functions, we're going to need to introduce the Heaviside theta function, or what some texts call the "unit step function." It is defined the following way:

$$\theta(t-c) = \begin{cases} 0, & -\infty \le t < c \\ 1, & c \le t < \infty \end{cases}$$

We can use this to represent piecewise functions. Let's try some functions:

$$\begin{aligned}
g(t) &= \begin{cases} 2t, & 0 \le t < 3 \\ e^{at}, & 3 \le t < \infty \end{cases} \\
&= 2t\theta(t) + \theta(t-3)\left[-2t + e^{at}\right]
\end{aligned}$$

Here we start the function at $2t$, and then at $t = 3$, we remove the $2t$ term and add the e^{at} term. If we are integrating $g(t)$ over a domain that starts at $t = 0$ (as we do for Laplace transforms), we can drop the $\theta(t)$ from the first term, giving us the following:

$$\int_0^\infty g(t)dt = \int_0^\infty \left(2t + \theta(t-3)\left[-2t + e^{at}\right]\right) dt$$

7.3.1 Laplace Transform of Heaviside Theta and Related Functions

This analysis is only important if we can actually transform the Heaviside theta function. We know that the Heaviside theta function is bounded (at 1), so it must be transformable. Let's perform the transformation:

$$\begin{aligned}\mathscr{L}\left\{\theta(t-c)\right\} &= \int_0^\infty e^{-st}\theta(t-c)dt \\ &= \int_0^c 0e^{-st}dt + \int_c^\infty e^{-st}dt\end{aligned}$$

To perform the above decomposition of $\theta(t-c)$, we need to assume $c \geq 0$. If $c < 0$, the theta function would have no effect on the integral.

$$\begin{aligned} &= \int_c^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\bigg|_{t=c}^{\infty} \\ \mathscr{L}\left\{\theta(t-c)\right\} &= \frac{1}{s}e^{-cs}; \quad c \geq 0 \end{aligned} \tag{7.5}$$

Using this, we can translate functions. Let's say we have a function $f(t)$ that is defined for $t \geq 0$, and we want to move it to the right 3 units, and we want to zero everything from 0 to 3. This can be represented the following way:

$$f(t) \xrightarrow{\text{3 units right}} \theta(t-3)f(t-3)$$

Let's try to find the Laplace transform for general rightward movement.

$$\begin{aligned}\mathscr{L}\left\{\theta(t-c)f(t-c)\right\} &= \int_0^c 0e^{-st}f(t-c)dt + \int_c^\infty e^{-st}f(t-c)dt \\ &= \int_c^\infty f(t-c)dt\end{aligned}$$

Now let's perform a u substitution: $u = t - c; \quad du = dt$.

$$\begin{aligned} &= \int_0^\infty e^{-s(u+c)}f(u)du \\ &= e^{-cs}\underbrace{\int_0^\infty e^{-su}f(u)du}_{\mathscr{L}\{f(t)\}} \\ \mathscr{L}\left\{\theta(t-c)f(t-c)\right\} &= e^{-cs}F(s) \end{aligned} \tag{7.6}$$

This result is very helpful, especially when inverting Laplace transforms. The operation

$$\mathscr{L}^{-1}\left\{F(s-b)\right\} = e^{bt}f(t)$$

also comes up very often. Let's prove this.

$$\begin{aligned}
\mathscr{L}\left\{e^{bt}f(t)\right\} &= \int_0^\infty e^{-st}e^{bt}f(t)dt \\
&= \int_0^\infty e^{(-s+b)t}f(t)dt
\end{aligned}$$

We can see that $f(t)$ can diverge only as fast as $e^{(b+a)t}$. Let's also now perform a substitution: $s' = s - b$.

$$\begin{aligned}
&= \int_0^\infty e^{-s't}f(t)dt \\
&= F(s') \\
\mathscr{L}\left\{e^{bt}f(t)\right\} &= F(s-b) \qquad (7.7)
\end{aligned}$$

7.3.2 ODEs with Discontinuous Forcing Functions

These two equations, (7.6) and (7.7), help us solve ODEs with discontinuous forcing functions. Let's do an example:

$$2y'' + y' + 2y = g(t) = \begin{cases} 0, & 0 \leq t < 5 \\ 1, & 5 \leq t < 20 \\ 0, & 20 \leq t < \infty \end{cases} \quad ; \quad y(0) = 0, y'(0) = 0$$

$$\begin{aligned}
\mathscr{L}\left\{2y'' + y' + 2y\right\} &= \mathscr{L}\left\{g(t)\right\} \\
&= \mathscr{L}\left\{\theta(t-5) - \theta(t-20)\right\} \\
&= \frac{1}{s}e^{-5s} - \frac{1}{s}e^{-20s} \\
F(s)\left[2s^2 + s + 2\right] &= \frac{1}{s}e^{-5s} - \frac{1}{s}e^{-20s} \\
F(s) &= \frac{1}{2s^2 + s + 2}\left[\frac{1}{s}e^{-5s} - \frac{1}{s}e^{-20s}\right] \\
&= \frac{e^{-5s} - e^{-20s}}{2s^3 + s^2 + 2s} \\
&= \left[e^{-5s} - e^{-20s}\right]H(s); \quad H(s) = \frac{1}{2s^3 + s^2 + 2s}
\end{aligned}$$

Let's just assume we know $h(t) = \mathscr{L}^{-1}\left\{H(s)\right\}$.

$$y(t) = \mathscr{L}^{-1}\left\{e^{-5s}H(s)\right\} - \mathscr{L}^{-1}\left\{e^{-20s}H(s)\right\}$$

Recall equation (7.6).

$$= \theta(t-5)h(t-5) - \theta(t-20)h(t-20) \qquad (7.8)$$

This result is what is known as the "formal solution," where $h(t)$ is assumed to be known. Now let's calculate $h(t)$.

$$h(t) = \mathscr{L}^{-1}\left\{H(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s(2s^2+s+2)}\right\}$$

$2s^2 + s + 2$ is an irreducible quadratic, so we need to put a linear term on top to perform partial fractions.[4] Let's now perform partial fractions.

$$\begin{aligned}
\frac{1}{s(2s^2+s+2)} &= \frac{A}{s} + \frac{Bs+C}{2s^2+s+2} \\
1 &= A(2s^2+s+2) + (Bs+C)(s) \\
&= s^2(2A+B) + s(A+C) + 2A \\
&\implies A = \frac{1}{2}, \quad C = -\frac{1}{2}, \quad B = -1 \\
\frac{1}{s(2s^2+s+2)} &= \frac{1}{2s} - \frac{s+\frac{1}{2}}{2s^2+1s+2} \\
&= \frac{1}{2s} - \frac{s+\frac{1}{2}}{2\left[s^2+\frac{1}{2}s+1\right]}
\end{aligned}$$

We need to convert this into a function we can take the inverse Laplace transform of. We will try to convert it into offset trigonometric functions by completing the square.

$$\begin{aligned}
&= \frac{1}{2s} - \frac{s+\frac{1}{2}}{2\left[(s+\frac{1}{4})^2 + \left(\frac{\sqrt{15}}{4}\right)^2\right]} \\
&= \frac{1}{2s} - \frac{1}{2}\left[\frac{(s+\frac{1}{4})+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right] \\
&= \frac{1}{2s} - \frac{1}{2}\left[\frac{(s+\frac{1}{4})}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right] \qquad (7.9)
\end{aligned}$$

[4] We could reduce this polynomial by factoring using its complex roots, but that procedure won't make this problem easier for us. It *is* helpful in certain situations, just not this one.

Let's tackle these two complicated terms one by one:

$$H_1(s) = \frac{(s+\frac{1}{4})}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

$$H_1(s') = \frac{s'}{s'^2 + \left(\frac{\sqrt{15}}{4}\right)^2}; \quad s' = s + \frac{1}{4}$$

$$\widetilde{H}_1(s) = H_1(s') = H_1\left(s + \frac{1}{4}\right)$$

Now let's recall the Laplace transform of $\cos(at)$.

$$\mathscr{L}\left\{\cos(at)\right\} = \frac{s}{s^2 + a^2}$$

This gives us the following relation:

$$\mathscr{L}^{-1}\left\{\widetilde{H}_1(s)\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right\} = \cos\left(\frac{\sqrt{15}}{4}t\right) = \tilde{h}_1(t)$$

Now let's recall the inverse Laplace transform for an offset $F(s)$, equation (7.7). This gives us the following relation:

$$\mathscr{L}^{-1}\left\{H_1(s)\right\} = h_1(t) = e^{-\frac{t}{4}}\mathscr{L}^{-1}\left\{\widetilde{H}_1(s)\right\} = e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}}{4}t\right)$$

Now let's tackle the second complicated term:

$$H_2(s) = \frac{\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

Let's first deal with the offset:

$$\widetilde{H}_2(s) = H_2\left(s + \frac{1}{4}\right) = \frac{\frac{1}{4}}{s^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

$$\implies h_2(t) = \mathscr{L}^{-1}\left\{H_2(s)\right\} = e^{-\frac{t}{4}}\tilde{h}_2(t)$$

Now recall the Laplace transform of $\sin(at)$:

$$\mathscr{L}\left\{\sin(at)\right\} = \frac{a}{s^2 + a^2}$$

When looking at our equation, we can see that $a = \frac{\sqrt{15}}{4}$. This means we need $\frac{\sqrt{15}}{4}$ in the numerator. This can be done by multiplying the top and bottom by $\sqrt{15}$.

$$\widetilde{H}_2(s) = \frac{1}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{s^2 + \left(\frac{\sqrt{15}}{4}\right)^2} \implies \tilde{h}_2(t) = \frac{1}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

This gives the final result for $\tilde{h}_2(t)$:

$$h_2(t) = \frac{1}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t\right)$$

Now we can put both $h_1(t)$ and $h_2(t)$ into their respective places when inverting equation (7.9). This gives us the following equation $h(t)$.

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\left\{\frac{1}{2s}\right\} - \frac{1}{2}\left[h_1(t) + h_2(t)\right] \\ &= \frac{1}{2} - \frac{1}{2} e^{\frac{-t}{4}} \left[\cos\left(\frac{\sqrt{15}}{4} t\right) + \frac{1}{\sqrt{15}} \sin\left(\frac{\sqrt{15}}{4} t\right)\right] \end{aligned}$$

Now we can plug this back into equation (7.8) to give our solution.

7.4 Adding Impulses with the Dirac Delta Function

What if we want to model a sudden hit to our system? We want a finite amount of energy to be put into the system while having that energy being added occur over 0 time. Let's call the function that models this behavior $\delta(t)$, where $\delta(t - t_0)$ adds an impulse our forcing function at time t_0. Let's say the integral of this function over all time is 1. How do we generate such a function? We make use of δ sequences. Let's first define our forcing function $g(t)$:

$$g(t) = d_\tau(t) = \begin{cases} \dfrac{1}{2\tau}, & -\tau < t < \tau \\ 0, & t < -\tau, \quad t > \tau \end{cases}$$

Integrating over all time should be 1:

$$\int_{-\infty}^{\infty} g(t)dt = \int_{-\infty}^{\infty} d_\tau(t)dt = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left.\frac{t}{2\tau}\right|_{t=-\tau}^{\tau} = 1$$

Notice that this result is independent of τ. To make this an impulse, we have to take the limit as τ goes to 0.

$$\lim_{\tau \to 0} \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = 1$$

Clearly, then, the following statements are true:

$$\begin{aligned} &\lim_{\tau \to 0} d_\tau(t) = 0, \quad t \neq 0 \\ &\lim_{\tau \to 0} d_\tau(0) \text{ is undefined.} \\ &\lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t)dt = 1 \end{aligned}$$

We can define a function $\delta(t)$ to have these properties.

$$\delta(t) = 0, \quad t \neq 0$$
$$\delta(0) \text{ is undefined}$$
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

We can also offset everything by t_0.

$$\delta(t - t_0) = 0, \quad t \neq t_0$$
$$\delta(t_0) \text{ is undefined}$$
$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1$$

Paul Dirac defined such a function for the work he was doing in Quantum Mechanics, so that is why this function is oftened referred to as the Dirac delta function. Let's try to do some math with this.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt &= \lim_{\tau \to 0} \int_{t_0 - \tau}^{t_0 + \tau} f(t) \frac{1}{2\tau} dt \\
&= \lim_{\tau \to 0} \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} f(t) dt
\end{aligned}$$

Let's define $F'(t) = f(t)$.

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \left[F(t_0 + \tau) - F(t_0 - \tau) \right]$$

Now we must employ L'Hopital's rule.

$$\begin{aligned}
&= \lim_{\tau \to 0} \frac{1}{2} \left[F'(t_0 + \tau) \frac{d}{d\tau} (t_0 + \tau) - F'(t_0 - \tau) \frac{d}{d\tau} (t_0 - \tau) \right] \\
&= \lim_{\tau \to 0} \frac{1}{2} \left[f(t_0 + \tau) + f(t_0 - \tau) \right] \\
&= \frac{f(t_0) + f(t_0)}{2} \\
&= f(t_0)
\end{aligned}$$

This result is very helpful, and makes subsequent uses of the delta function more easy to deal with. One exemplar subsequent use would be to find the Laplace transform of the delta function.

$$\mathscr{L} \{ \delta(t - t_0) \} = \int_{0}^{\infty} e^{-st} \delta(t - t_0) dt$$

Here we must assume that $t_0 > 0$. If we didn't the integral would just evaluate to 0. Once we make this assumption, we can just use the result we just calculated.

$$\mathscr{L}\left\{\delta(t-t_0)\right\} = e^{-st_0}$$

We can also quickly get the Laplace transform of the product of the delta function with any function.

$$\mathscr{L}\left\{\delta(t-t_0)g(t)\right\} = e^{-st_0}g(t_0)$$

7.4.1 Drama on the Playground

Let's suppose you've been tasked with taking care of your younger sister. Maybe she's a little grumpy, so you walk her to your local elementary school. In the playground, she hops on the swing, and you give it a little bit a push. She gets scared immediately and wants to get off, so you stop her exactly one half period after you pushed her to start. We can model this movement with the following IVP. For simplicity, we will assume the swing is frictionless ($\gamma = 0$).

$$y'' + y = \delta(t-\pi); \quad y(0) = 0, y'(0) = 1$$

Here, gravity acts as the "spring force." To solve this, we will transform both sides of the equation to find $F(s)$.

$$\begin{aligned}
\mathscr{L}\left\{y'' + y\right\} &= \mathscr{L}\left\{\delta(t-\pi)\right\} \\
F(s)[s^2+1] - 1 &= e^{-s\pi} \\
F(s) &= \frac{1+e^{-s\pi}}{s^2+1} \\
&= \frac{1}{s^2+1} + e^{-s\pi}\left(\frac{1}{s^2+1}\right) \\
\implies y(t) &= \sin(t) + \theta(t-\pi)\sin(t-\pi) \\
&= \sin(t) - \theta(t-\pi)\sin(t)
\end{aligned}$$

Here we see that we go through one lobe of the sine wave, one half period. After that, there is no movement.

7.5 The Convolution Integral

Let's suppose we have a function we want to take the inverse Laplace transform of. Let's suppose this function is a product of two functions in the Laplace domain.

$$H(s) = F(s)G(s)$$

$$F(s) = \int_0^\infty e^{-st}f(t)dt \quad G(s) = \int_0^\infty e^{-st}g(t)dt$$

That means we can write an expression for $H(s)$:

$$H(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau \int_0^\infty e^{-s\xi} g(\xi) d\xi$$

We can simplify this equation.

$$= \int_0^\infty f(\tau) \left[\int_0^\infty e^{-s(\tau+\xi)} g(\xi) d\xi \right] d\tau$$

Now we make the substitution $t = \tau + \xi$.

$$= \int_0^\infty f(\tau) \left[\int_\tau^\infty e^{-st} g(t-\tau) dt \right] d\tau$$

Now we swap the order of integration.

$$\begin{aligned} &= \int_0^\infty e^{-st} \left[\int_0^t f(\tau) g(t-\tau) d\tau \right] dt \\ &= \mathscr{L} \left\{ \int_0^t f(\tau) g(t-\tau) d\tau \right\} \qquad (7.10) \end{aligned}$$

Here we introduce the convolution operator.

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

We can check to see that this operator is commutative:

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

Introduce $\xi = t - \tau$.

$$\begin{aligned} &= -\int_t^0 f(t-\xi) g(\xi) d\xi \\ &= \int_0^t g(\xi) f(t-\xi) d\xi \\ &= g * f \end{aligned}$$

We can plug this operator back into equation (7.10) to get the following:

$$H(s) = \mathscr{L} \{ f * g \}$$

This implies the following result:

$$\mathscr{L} \{ f * g \} = F(s)G(s); \quad \mathscr{L}^{-1} \{ F(s)G(s) \} = f * g$$

We can use this result to solve more differential equations. Let's look at an example.

$$h(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}$$

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \\ g(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t} \end{aligned}$$

We recognize that $h(t)$ is just the convolution of $f(t)$ and $g(t)$.

$$\begin{aligned} h(t) &= e^{2t} * e^{3t} \\ &= \int_0^t e^{3(t-\tau)} e^{2\tau} d\tau \\ &= e^{3t} \int_0^t e^{-\tau} d\tau = e^{3t}\left(e^{-t}\right)\Big|_{\tau=0}^{t} \\ &= e^{3t}\left[1 - e^{-t}\right] \\ &= e^{3t} - e^{2t} \end{aligned}$$

We could have achieved the same result using partial fractions.

Let's do an example where the convolution integral is actually helpful. Suppose you are working for a phone manufacturer, and you really don't want your phones to explode. In your attempts to model the heat flow of the phone designs, you get the following ODE where you're waiting for workers elsewhere to provide a function $g(t)$. Because you're a productive worker, you want to solve as much as we can without specifying $g(t)$.

$$y'' + \omega_0^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$$

Transforming both sides:

$$\begin{aligned} F(s)\left[s^2 + \omega_0^2\right] - 1 &= \mathscr{L}\left\{g(t)\right\} = G(s) \\ F(s) &= \frac{G(s)+1)}{s^2+\omega_0^2} \\ &= \frac{1}{s^2+\omega_0^2} + G(s)\frac{1}{s^2+\omega_0^2} \\ &= \frac{1}{\omega_0}\left[\mathscr{L}\left\{\sin(\omega_0 t)\right\} + \mathscr{L}\left\{g(t) * \sin(\omega_0 t)\right\}\right] \\ \implies y(t) &= \frac{1}{\omega_0}\left[\sin(\omega_0 t) + \int_0^t g(t-\tau)\sin(\omega_0 \tau) d\tau\right] \end{aligned}$$

Let's say the engineers you were waiting for gave you four functions to try:

$$g(t) = \sin(\omega t), \quad \omega \neq \omega_0$$
$$g(t) = 10^{12} e^{1015t}$$
$$g(t) = 10^{-5} e^{10^{-21} t^2}$$
$$g(t) = \frac{6}{t-6} e^{3t}$$

You're on a tight schedule, so you quickly tell your group of engineers to evaluate the convolution integrals you have prepared. Immediately your boss tells you that your services are no longer needed, and you are walked out of your workplace. What did you do wrong? When we used the convolution integral, we assumed that $\mathscr{L}\{g(t)\}$ exists. However, we didn't check the functions to see if their Laplace transforms existed. You can see that the third and fourth equations diverge faster than e^{at}.

7.5.1 Integro-Differential Equations

Integro-differential equations are equations containing both derivatives and integrals. We can solve certain types of integro-differential equations with the convolution integral. Let's consider this example:

$$y' = cos(t) + \int_0^t y(\tau)\cos(t-\tau)d\tau, \quad y(0) = 1$$

Let's take the Laplace transform of both sides.

$$\begin{aligned}
\mathscr{L}\{y'\} = sF(s) - 1 &= \mathscr{L}\{\cos(t)\} + \mathscr{L}\left\{\int_0^t y(\tau)\cos(t-\tau)d\tau\right\} \\
&= \frac{s}{s^2+1} + \mathscr{L}\{y(t)\}\,\mathscr{L}\{\cos(t)\} \\
&= \frac{s}{s^2+1}\left[1 + F(s)\right] \\
\implies \left[s - \frac{s}{s^2+1}\right] F(s) &= \frac{s}{s^2+1} + 1 \\
\left[s(s^2+1) - s\right] F(s) &= s + s^2 + 1 \\
s^3 F(s) &= s^2 + s + 1 \\
F(s) &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} \\
\implies y(t) &= 1 + t + \frac{1}{2}t^2
\end{aligned}$$

8 Systems of First-Order Linear ODEs

8.1 System Conversion

We can convert a system of n first-order ODEs into one n^{th}-order ODE.